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University of Technology and Applied Sciences
Sultanate of Oman
GENERAL FOUNDATION PROGRAM

# Mathematics for Sciences




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University of Technology and Applied Sciences

Sultanate of Oman

General Foundation Program

# Mathematics for Sciences

## FPMS0001

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# Mathematics for Sciences-Learning Outcomes

| | |
|---|---|
| a) | Describe the set of real numbers, all its subsets, and their relationship. |
| b) | Identify and use the arithmetic properties of subsets of integers, rational, irrational, and real numbers, including closure properties for the four basic arithmetic operations where applicable. |
| c) | Demonstrate an understanding of the exponent laws, and apply them to simplify expressions and manipulate fractions, ratios, decimals, and percentages. |
| d) | Understand measurements and conversion from one unit to another. |
| e) | Translate worded problems into mathematical expressions and model simple real-life problems with equations and inequalities. |
| f) | Use the coordinate plane to solve algebraic and geometric problems, and understand geometric concepts such as the equation of a circle, perpendicular, parallel, and tangent lines. |
| g) | Use the three types of symmetry of an equation to sketch its graph. |
| h) | Know the relationship between degree and radian measure of an angle and find the length of a circular arc and the area of a sector. |
| i) | Understand trigonometric and circular functions and use the fundamental trigonometric identities in various problems. |
| j) | Solve right-angled triangles using angles of elevation and depression. |
| k) | Apply knowledge of basic algebra and trigonometry in real-life problems. |
| l) | Simplify rational expressions and rationalize numerators or denominators. |
| m) | Solve linear equations, equations involving radicals, fractional expressions, and inequalities. |
| n) | Perform operations on polynomials and manipulate numerical and polynomial expressions and solve first-degree equations. |
| o) | Use the quadratic formula to find the roots of a second-degree polynomial. |
| p) | Demonstrate an understanding of the definition of a function and its graph. |
| q) | Solve quadratic equations using quadratic formula. |
| r) | Demonstrate an understanding of trigonometric identities. |
| s) | Define and manipulate exponential and logarithmic functions and solve problems from real life applications. |
| t) | Understand the inverse relationship between exponents and logarithms and use this relationship to solve related problems. |
| u) | Understand the definition of the different types of angles and measure them in degrees and radians. |
| v) | Describe analytically the trigonometric and circular functions. |
| w) | Use the law of sine and cosines to solve a triangle and real-life problems. |
| x) | Use appropriate software to interpret equations and graphs. |
| y) | Understand basic concepts of descriptive statistics, mean, median, mode and summarize data into tables and simple graphs (bar charts, histogram and pie chart). |
| z) | Understand basic probability concepts and compute the probability of simple events using tree diagrams and formulas for permutations and combinations. |

# Module I

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# Chapter 1

# REAL NUMBERS

## Contents


**1.1 Classification of Real Numbers**

1.1.1 Identify Counting Numbers and Whole Numbers

1.1.2 Rational and Irrational Numbers

**1.2 Properties of Real Numbers**

**1.3 Fractions, Decimals, Ratios and Percent**

1.3.1 Fractions

1.3.2 Decimals

1.3.3 Ratio and Percent


**Learning outcome covered:**

**a.** Describe the set of real numbers, all its subsets and their relationship.

**b.** Identify and use the arithmetic properties of subsets of integers, rational, irrational, and real numbers, including closure properties for the four basic arithmetic operations where applicable.

**c.** Manipulate fractions, ratios, decimals, and percentages.

**k.** Apply knowledge of basic algebra and trigonometry in real life problems.

### Learning Objectives

**By the end of this chapter, the students will be able to:**

- Classify different types of real numbers
- Use the commutative and associative properties
- Simplify expressions using the distributive property
- Recognize and use the identity and inverse properties of addition and multiplication
- Use the properties of zero

- Simplify expressions using the properties of identities, inverses, and zero
- Add and Subtract Fractions
- Convert percents to fractions and decimals
- Convert decimals and fractions to percents

# Introduction

Even though counting is first taught at a young age, mastering mathematics, which is the study of numbers, requires constant attention. If it has been a while since you have studied math, it can be helpful to review basic topics. In this chapter, we will focus on numbers as well as four arithmetic operationsaddition, subtraction, multiplication and division. We will also discuss some vocabulary that we will use throughout this book.

## 1.1 Classification of Real Numbers

Algebra uses numbers and symbols to represent words and ideas. Let us look at the numbers first. In this section we describe the set of real numbers, all its subsets and their relationship.

### 1.1.1 Identify Counting Numbers and Whole Numbers

The most basic numbers used in algebra are those we use to count objects: 1, 2, 3, 4, 5, and so on. These are called the **counting numbers**. Counting numbers are also called **natural numbers**.

**Counting Numbers or Natural Numbers(N)**

The counting numbers start with 1 and continue.

$$N = \{1, 2, 3, 4, 5, \cdots\}$$

The discovery of the number zero was a big step in the history of mathematics. Including zero with the counting numbers gives a new set of numbers called the **whole numbers**.

**Whole Numbers (W)**

The whole numbers are the counting numbers and zero.

$$W = \{0, 1, 2, 3, 4, 5, \cdots\}$$

#### Introduction to Integers

A negative number is a number that is less than 0. Both positive and negative numbers can be represented on a number line. We could write a plus sign, +, before a positive number such as +2 or +3, but it is customary to omit the plus sign and write only the number. If there is no sign, the number is assumed to be positive.

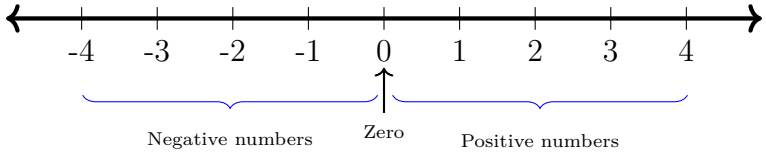


**Figure 1**

The arrows at either end of the line indicate that the number line extends forever in each direction. There is no greatest positive number and there is no smallest negative number.

**Integers**

The set of counting numbers, their opposites, and 0 is the set of integers.

**Integers (Z)**

The set of counting numbers, their opposites, and 0 is the set of integers.

$$Z = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$$

### 1.1.2 Rational and Irrational Numbers

**Rational Numbers**

**Rational Numbers (Q)**

A rational number is a number that can be written in the form $\frac{p}{q}$, where $p$ and $q$ are integers and $q \neq 0$.

$$Q = \left\{ \frac{p}{q} : p \text{ and } q \text{ are integers and } q \neq 0 \right\}$$

Are integers rational numbers? To decide if an integer is a rational number, we try to write it as a ratio of two integers. An easy way to do this is to write it as a fraction with denominator one. $3 = \frac{3}{1}$, $-8 = \frac{-8}{1}$ and $0 = \frac{0}{1}$. Since any integer can be written as the ratio of two integers, all integers are rational numbers. Remember that all the counting numbers and all the whole numbers are also integers, and so they, too, are rational.

What about decimals? Are they rational? Think about the decimal 7.3. We can write it as $\frac{73}{10}$. It is a rational number.

Every rational number can be written both as a ratio of integers and as a decimal that either **stops** [e.g., 4.275] **or repeats** [e.g., 2.757575...]. The table below shows the numbers we looked at expressed as a ratio of integers and as a decimal.

| Rational Numbers | | |
|---|---|---|
| | Fractions | Integers |
| Number | $\frac{4}{5}, -\frac{7}{8}, \frac{13}{4}, \frac{-20}{3}$ | $-2, -1, 0, 1, 2, 3$ |
| Ratio of Integer | $\frac{4}{5}, \frac{-7}{8}, \frac{13}{4}, -\frac{20}{3}$ | $\frac{-2}{1}, \frac{-1}{1}, \frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}$ |
| Decimal number | $0.8, -0.875, 3.25, -6.\overline{6}\,[-6.666\cdots]$ | $-2.0, -1.0, 0.0, 1.0, 2.0, 3.0$ |

### Irrational Numbers

Are there any decimals that do not stop or repeat? Yes. The number $\pi$ (the Greek letter $\pi$, pronounced pi), which is very important in describing circles, has a decimal form that does not stop or repeat.

$$\pi = 3.141592654\cdots$$

Similarly, the decimal representations of square roots of numbers that are not perfect squares never stop and never repeat. For example,

$$\sqrt{5} = 2.236067978\cdots$$

A decimal that does not stop and does not repeat cannot be written as the ratio of integers. We call this kind of number an irrational number.

**Irrational Numbers (I)**

An irrational number is a number that cannot be written as the ratio of two integers. Its decimal form does not stop and does not repeat.

**Example 1.1:** Identify each of the following as rational or irrational:

**a.** $0.58\overline{3}$ **b.** $0.475$ **c.** $3.605551275...$

**Solution:**

(a) The bar above the 3 indicates that it repeats. Therefore, $0.58\overline{3} = 0.583333...$ is a repeating decimal, and is therefore a rational number.

(b) This decimal stops after the 5 , so it is a rational number.

(c) The ellipsis (...) means that this number does not stop. There is no repeating pattern of digits. Since the number doesn't stop and doesn't repeat, it is irrational.

---

Let us think about square roots now. Square roots of perfect squares are always whole numbers, so they are rational. But the decimal forms of square roots of numbers that are not perfect squares never stop and never repeat, so these square roots are irrational.

**Example 1.2:** Identify each of the following as rational or irrational:

**a.** $\sqrt{36}$ **b.** $\sqrt{44}$

**Solution:**

(a) The number 36 is a perfect square, since $6^2 = 36$. So $\sqrt{36} = 6$. Therefore $\sqrt{36}$ is rational.

(b) Remember that $6^2 = 36$ and $7^2 = 49$, so 44 is not a perfect square. This means $\sqrt{44}$ is irrational.

## Classify Real Numbers

We have seen that all counting numbers are whole numbers, all whole numbers are integers, and all integers are rational numbers. Irrational numbers are a separate category of their own. When we put together the rational numbers and the irrational numbers, we get the set of real numbers.

Figure 2 illustrates how the number sets are related.

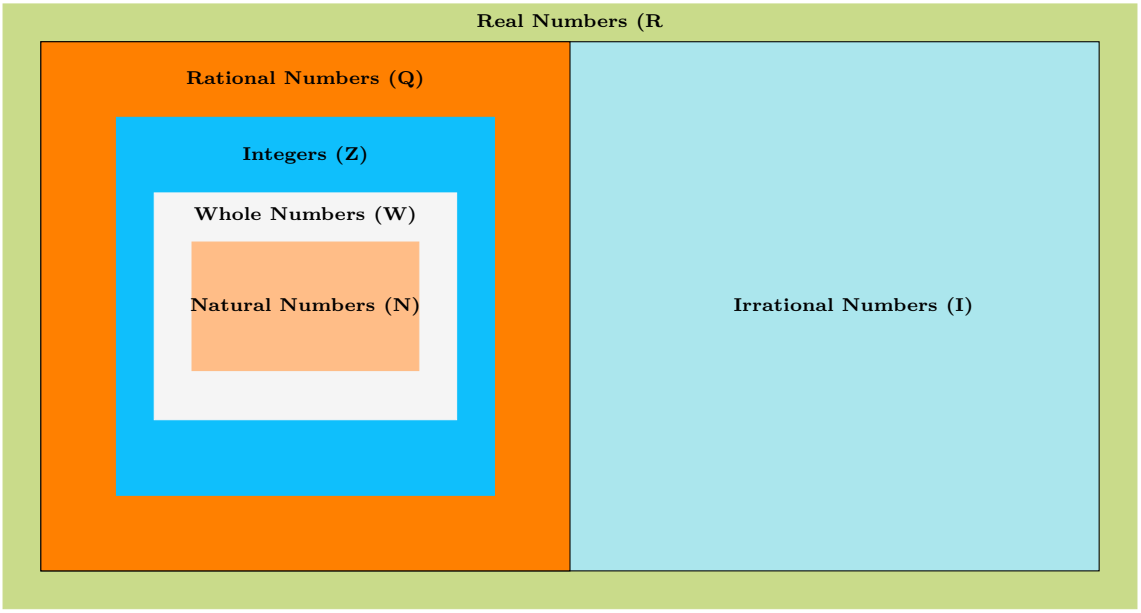



**Figure 2**

### Real Numbers (R)

Real numbers are numbers that are either rational or irrational.

**Example 1.3:** Determine whether each of the numbers in the following list is a (a) whole number, (b) integer, (c) rational number, (d) irrational number, and (e) real number.

$$\text{-7},\ \frac{14}{5},\ 8,\ \sqrt{5},\ 5.9,\ -\sqrt{64}$$

**Solution:**

(a) The whole numbers are 0, 1, 2, 3, The number 8 is the only whole number given.

(b) The integers are the whole numbers, their opposites, and 0. From the given numbers, 7 and 8 are integers. Also, notice that 64 is the square of 8 so $-\sqrt{64} = -8$. So the integers are 7, 8, $-\sqrt{64}$.

(c) Since all integers are rational, the numbers 7, 8, and $-\sqrt{64}$ are also rational. Rational numbers also include fractions and decimals that terminate or repeat, so $\frac{14}{5}$ and 5.9 are rational.

(d) The number 5 is not a perfect square, so $\sqrt{5}$ is irrational.

(e) All of the numbers listed are real.

We'll summarize the results in a table.

| Number | Whole | Integer | Rational | Irrational | Real |
|---|---|---|---|---|---|
| -7 | | ✓ | ✓ | | ✓ |
| $\frac{14}{5}$ | | | ✓ | | ✓ |
| 8 | ✓ | ✓ | ✓ | | ✓ |
| $\sqrt{5}$ | | | | ✓ | ✓ |
| 5.9 | | | ✓ | | ✓ |
| $-\sqrt{64}$ | | ✓ | ✓ | | ✓ |

## 1.1 Section Exercises

### Rational Numbers

*In the following exercises, determine which of the given numbers are rational and which are irrational.*

**1.** $0.75, 0.22\overline{3}, 1.39174\cdots$ **2.** $0.36, 0.94729\cdots, 2.52\overline{8}$ **3.** $0.4\overline{5}, 1.919293\cdots, 3.59$

*In the following exercises, identify whether each number is rational or irrational.*

**4.** **a.** $\sqrt{25}$ **b.** $\sqrt{30}$

**5.** **a.** $\sqrt{44}$ **b.** $\sqrt{49}$

**6.** **a.** $\sqrt{164}$ **b.** $\sqrt{169}$

### Classifying Real Numbers

*In the following exercises, determine whether each number is whole, integer, rational, irrational, and real.*

**7.** $-8, 0, 1.95286...., \frac{12}{5}, \sqrt{36}, 9$

**8.** $-9, -3\frac{4}{9}, -\sqrt{9}, 0.4\overline{09}, \frac{11}{6}, 7$

## 1.2 Properties of Real Numbers

Addition (+) and Multiplication ($\times$ or $\cdot$) are two important operations defined on the set of all real numbers. In this section, we will take a look at some properties of real numbers with respect to these two operations.

### 1. Closure Properties

We know the sum of any two real numbers is again a real number and also the product of any two real numbers is also a real number. These properties are given below.

**Closure Properties**

**Closure Property of Addition:** If $a$ and $b$ are real numbers, then $a + b$ is also a real number.

**Closure Property of Multiplication:** If $a$ and $b$ are real numbers, then $a \cdot b$ is also a real number.

### 2. Commutative Properties

**Commutative Properties**

**Commutative Property of Addition:** if $a$ and $b$ are real numbers, then

$$a + b = b + a$$

**Commutative Property of Multiplication:** if $a$ and $b$ are real numbers, then

$$a \cdot b = b \cdot a$$

**Example 1.4:** Use the commutative properties to rewrite the following expressions:

**a.** $-1 + 3 =$ ____ **b.** $4 \cdot 9 =$ ____

**Solution:**

**a.** $-1 + 3 = 3 + (-1)$ Use the commutative property of addition to change the order.

**b.** $4 \cdot 9 = 9 \cdot 4$ Use the commutative property of multiplication to change the order.

### 3. Associative Properties

**Associative Properties**

**Associative Property of Addition:** if $a$, $b$, and $c$ are real numbers, then

$$(a + b) + c = a + (b + c)$$

**Associative Property of Multiplication:** if $a$, $b$, and $c$ are real numbers, then

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

**Example 1.5:** Use the associative properties to rewrite the following expressions:

**a.** $(3 + 0.6) + 0.4 =$ ____ **b.** $\left(-4 \cdot \frac{2}{5}\right) \cdot 15 =$ ____

**Solution:**

**a.** $(3 + 0.6) + 0.4 = 3+(0.6+0.4)$ Change the grouping.

*Notice that 0.6 + 0.4 is 1, so the addition will be easier if we group as shown on the right.*

**b.** $\left(-4 \cdot \frac{2}{5}\right) \cdot 15 = -4 \cdot \left(\frac{2}{5} \cdot 15\right)$ Change the grouping.

*Notice that* $\frac{2}{5} \cdot 15$ *is 6. The multiplication will be easier if we group as shown on the right.*

---

### 4. Distributive Property

In algebra, we use the Distributive Property to remove parentheses as we simplify expressions.

**Distributive Property**

If $a$, $b$, and $c$ are real numbers, then $a(b + c) = a \cdot b + a \cdot c$

**Example 1.6:** Simplify: $3(x + 4)$

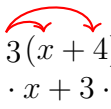
**Solution:**

$3(x + 4) = 3 \cdot x + 3 \cdot 4$ Distribute.

$= 3x + 12$ Multiply.

---

***Some students find it helpful to draw in arrows to remind them how to use the Distributive Property. Then the first step in the Example would look like this:***




**Example 1.7:** Simplify: $-11(4-3a)$

**Solution:**

$-11(4-3a) = -11 \cdot 4 - (-11) \cdot 3a$ Distribute.

$= -44 - (-33a)$ Multiply.

$= -44 + 33a$ Simplify.

You could also write the result as $33a-44$. Do you know why?

---

### 5. Identity Properties of Addition and Multiplication

What happens when we add zero to any number? Adding zero doesnt change the value. For this reason, we call 0 the **additive identity**.

What happens when you multiply any number by one? Multiplying by one doesnt change the value. So we call 1 the **multiplicative identity**.

**Identity Properties**

The **identity property of addition:** for any real number $a$,

$$a + 0 = a \qquad 0 + a = a$$

0 is called the **additive identity**

The **identity property of multiplication**: for any real number $a$,

$$a \cdot 1 = a \qquad 1 \cdot a = a$$

1 is called the **multiplicative identity**

**Example 1.8:** Identify whether each equation demonstrates the identity property of addition or multiplication.

**a.** $7 + 0 = 7$ **b.** $-16(1) = -16$

**Solution:**

**a.** We are adding 0. We are using the identity property of addition.

**b.** We are multiplying by 1. We are using the identity property of multiplication.

### 6. Inverse Properties of Addition and Multiplication

Well formally state the Inverse Properties here:

**Inverse Properties**

**Inverse Property of Addition** for any real number $a$,

$$a + (-a) = 0$$

$-a$ is the **additive inverse** of $a$.

**Inverse Property of Multiplication** for any real number $a \neq 0$,

$$a \cdot \frac{1}{a} = 1$$

$\frac{1}{a}$ is the **multiplicative inverse** of $a$.

**Example 1.9:** Find the additive inverse of each expression:

**a.** 13 **b.** $-\frac{5}{8}$ **c.** 0.6

**Solution:** To find the additive inverse, we find the opposite.

(a) The additive inverse of 13 is its opposite, $-13$.

(b) The additive inverse of $-\frac{5}{8}$ is its opposite, $\frac{5}{8}$.

(c) The additive inverse of 0.6 is its opposite, $-0.6$.

**Example 1.10:** Find the multiplicative inverse:

**a.** 9 **b.** $-\frac{1}{9}$ **c.** 0.9

**Solution:** To find the multiplicative inverse, we find the reciprocal.

(a) The multiplicative inverse of 9 is its reciprocal, $\frac{1}{9}$.

(b) The multiplicative inverse of $-\frac{1}{9}$ is its reciprocal, $-9$.

(c) To find the multiplicative inverse of 0.9, we first convert 0.9 to a fraction, $\frac{9}{10}$. Then we find the reciprocal, $\frac{10}{9}$.

## 7. Properties of Zero

We have already learned that zero is the additive identity, since it can be added to any number without changing the numbers identity. But zero also has some special properties when it comes to multiplication and division.

*What happens when you multiply a number by 0?* Multiplying by 0 makes the product equal zero.

*What about dividing with 0?* Think about a real example: if there are no cookies in the cookie jar and three people want to share them, how many cookies would each person get? There are 0 cookies to share, so each person gets 0 cookies. $0 \div 3 = 0$.

*Now lets think about dividing a number by zero.* What is the result of dividing 4 by 0? Is there a number that multiplied by 0 gives 4? There is no real number that can be multiplied by 0 to obtain 4. We can conclude that there is no answer to $4 \div 0$ and is **undefined**.

### Properties of Zero

**Multiplication by Zero**

- For any real number $a$, $a \cdot 0 = 0 \qquad 0 \cdot a = 0$

**Division with Zero**

- For any real number $a \neq 0$, $\frac{0}{a} = 0$.
- For any real number $a$, $\frac{a}{0}$ is undefined.

**Example 1.11:** **Simplify: a.** $-25 \times 0$ **b.** $\frac{0}{10}$ **c.** $\frac{32}{0}$

**Solution:** To find the multiplicative inverse, we find the reciprocal.

(a) $-25 \times 0 = 0$ (b) $\frac{0}{10} = 0$ (c) $\frac{32}{0}$ is undefined.

All the properties of real numbers we have used in this chapter are summarized in Table 1.

| Property | of Addition | of Multiplication |
|---|---|---|
| **Closure Property**<br>If $a$ and $b$ are real numbers then... | $a + b$ is also a real number | $a \cdot b$ is also a real number |
| **Commutative Property**<br>If $a$ and $b$ are real numbers then... | $a + b = b + a$ | $a \cdot b = b \cdot a$ |
| **Associative Property**<br>If $a$, $b$ and $c$ are real numbers then... | $a + (b + c) = (a + b) + c$ | $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ |
| **Identity Property**<br>For any real number $a$, | 0 is the additive identity<br>$a + 0 = a$<br>$0 + a = a$ | 1 is the multiplicative identity<br>$a \cdot 1 = a$<br>$1 \cdot a = a$ |
| **Inverse Property**<br>For any real number $a$, | $-a$ is the additive inverse of $a$<br>$a + (-a) = 0$ | For $a \neq 0$<br>$\frac{1}{a}$ is the multiplicative inverse of $a$<br>$a \cdot \frac{1}{a} = 1$ |
| **Distributive Property** | If $a$, $b$, $c$ are real numbers, then $a \cdot (b + c) = a \cdot b + a \cdot c$ | |
| **Properties of Zero**<br>For any real number $a$, | $a \cdot 0 = 0$ and $0 \cdot a = 0$ | |
| For any real number $a$, $a \neq 0$ | $\frac{0}{a} = 0$<br>$\frac{a}{0}$ is undefined. | |

**Table 1:** Properties of Real Numbers

## 1.2 Section Exercises

**Use the Commutative and Associative Properties** *In the following exercises, use the commutative properties to rewrite the given expression.*

**1.** $8+9=$ ____ **2.** $7+6=$ ____ **3.** $8(-12)=$ ____

**4.** $y+1=$ ____ **5.** $-2a=$ ____ **6.** $-3m=$ ____

*In the following exercises, use the associative properties to rewrite the given expression.*

**7.** $(11+9)+14=$ ____ **8.** $(21+14)+9=$ ____ **9.** $(12\cdot 5)\cdot 7=$ ____

**10.** $(7\cdot 6)\cdot 9=$ ____ **11.** $(-7+9)+8=$ ____ **12.** $3(4x)=$ ____

*Simplify Expressions Using the Commutative and Associative Properties*

**13.** $-45a+15+45a$ **14.** $9y+23+(-9y)$ **15.** $\frac{1}{2}+\frac{7}{8}+\left(-\frac{1}{2}\right)$

**16.** $\left(\frac{5}{6}+\frac{8}{15}\right)+\frac{7}{15}$ **17.** $\left(\frac{1}{12}+\frac{4}{9}\right)+\frac{5}{9}$ **18.** $14x+19y+25x+3y$

**Simplify Expressions Using the Distributive Property**

*In the following exercises, simplify using the distributive property.*

**19.** $4(x+8)$ **20.** $3(a+9)$ **21.** $8(4y+9)$

**22.** $7(3p-8)$ **23.** $5(7u-4)$ **24.** $\frac{1}{2}(n+8)$

**25.** $(y+4)p$ **26.** $-3(a+11)$ **27.** $-(r+7)$

**Recognize the Identity Properties of Addition and Multiplication**

*In the following exercises, identify whether each example is using the identity property of addition or multiplication.*

**28.** $101+0=101$ **29.** $\frac{3}{5}(1)=\frac{3}{5}$ **30.** $-9\cdot 1=-9$

**Use the Inverse Properties of Addition and Multiplication**

*In the following exercises, find the additive and multiplicative inverse.*

**31.** 8 **32.** 14 **33.** $-17$

**34.** 0.5 **35.** $\frac{7}{12}$ **36.** $\frac{8}{13}$

**Use the Properties of Zero**

*In the following exercises, simplify using the properties of zero.*

**37.** $48\cdot 0$ **38.** $0\div\frac{11}{12}$ **39.** $\dfrac{3}{0}$

**40.** $\dfrac{0}{24}$ **41.** $5.72\div 0$ **42.** $\dfrac{\frac{1}{10}}{0}$

## 1.3 Fractions, Decimals, Ratios and Percent

In this section, we will learn about numbers that describe parts of a whole. These numbers, called fractions, are very useful both in algebra and in everyday life. Also we will explore decimals, ratios and percentage in this section.

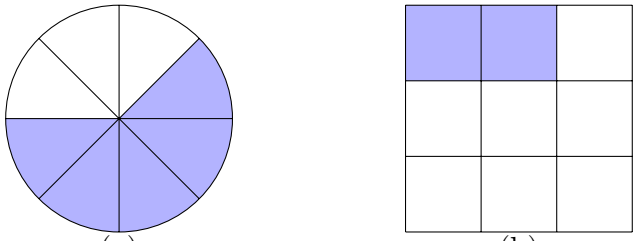
### 1.3.1 Fractions

A fraction is a way to represent parts of a whole. The denominator $b$ represents the number of equal parts the whole has been divided into, and the numerator $a$ represents how many parts are included. The denominator, $b$, cannot equal zero because division by zero is undefined.

**Fractions**

A fraction is written $\frac{a}{b}$, where $a$ and $b$ are integers and $b \neq 0$. In a fraction, $a$ is called the **numerator** and $b$ is called the **denominator**.

**Example 1.12:** Name the fraction of the shape that is shaded in each of the figures.



(a) (b)

**Solution:** We need to ask two questions. First, how many equal parts are there? This will be the denominator. Second, of these equal parts, how many are shaded? This will be the numerator.

**a.** Five out of eight parts are shaded. Therefore, the fraction of the circle shaded is $\frac{5}{8}$.

**b.** Two out of nine parts are shaded. Therefore, the fraction of the square shaded is $\frac{2}{9}$.

.....

**Improper Fractions and Mixed Numbers**

**Proper and Improper Fractions**

The fraction $\frac{a}{b}$ is a proper fraction if $a < b$ and an improper fraction if $a \geq b$.

An improper fraction can be converted into mixed number and vice versa.

The improper fraction $\frac{8}{5}$ is one whole, 1, plus three fifths, $\frac{3}{5}$, or $1\frac{3}{5}$, which is read as *one*

*and three-fifths*. The number $a\frac{b}{c}$ is called a **mixed number**. A mixed number consists of a whole number and a fraction.

### Convert between Improper Fractions and Mixed Numbers

***How To* :: Convert an Improper Fraction to a Mixed number**




Step 1. Divide the denominator by the numerator.

Step 2. Identify the quotient, remainder, and divisor.

Step 3. Write the mixed number as $Quotient\frac{remainder}{divisor}$

**Example 1.13:** Convert the improper fraction $\frac{33}{8}$ to a mixed number.

**Solution:** Divide the denominator into the numerator. Identify the quotient, remainder, and divisor.

Quotient $= 4$ and Remainder $= 1$ $\qquad\qquad \frac{33}{8}$

$\frac{33}{8} = 4\frac{1}{8}$

.....

***How To* :: Convert a Mixed number to an Improper Fraction**




Step 1. Multiply the whole number by the denominator.

Step 2. Add the numerator to the product found in Step 1.

Step 3. Write the final sum over the original denominator.

**Example 1.14:** Convert the mixed number $4\frac{2}{3}$ to an improper fraction.

**Solution:** $4\frac{2}{3} = \frac{4\times 3+2}{3} = \frac{14}{3}$

.....

## Manipulation of fractions

Let us begin with some important concepts which are essential for addition and multiplication of fractions.

### Multiples and Factors

A multiple of a number is the product of the number and a counting number. 3, 6, 9, 12, $\cdots$ are the multiples of 3.

**Multiple of a Number**

A number is a multiple of $n$ if it is the product of a counting number and $n$.

Another way to say that 20 is a multiple of 4 is to say that 20 is divisible by 4. In fact, $20 \div 4$ is 5, so 20 is $5 \times 4$. Notice that 18 is not a multiple of 4. When we divided 18 by 4 we did not get a counting number, so 18 is not divisible by 4.

**Divisibility**

If a number $m$ is a multiple of $n$, then we say that $m$ is divisible by $n$.

### Factors of a Number

We know that 72 is the product of 8 and 9, so we can say 72 is a multiple of 8 and 72 is a multiple of 9. Another way to talk about this is to say that 8 and 9 are factors of 72.

**Factors**

If $a \cdot b = m$, then $a$ and $b$ are factors of $m$, and $m$ is the product of $a$ and $b$.

### How to find all the factors of any Counting number?

**Step 1.** Divide the number by each of the counting numbers, in order, until the quotient is smaller than the divisor.

- If the quotient is a counting number, the divisor and quotient are a pair of factors.
- If the quotient is not a counting number, the divisor is not a factor.

**Step 2.** List all the factor pairs.

**Step 3.** Write all the factors in order from smallest to largest.

### Prime and Composite Numbers

Some numbers, like 72, have many factors. Other numbers, such as 7, have only two factors: 1 and the number. A number with **only two factors** is called a prime number. A number with more than two factors is called a composite number. The number 1 is neither prime nor composite. It has only one factor, itself.

**Prime Numbers and Composite Numbers**

A prime number is a counting number greater than 1 whose only factors are 1 and itself.

A composite number is a counting number that is not prime.

The prime numbers less than 20 are 2, 3, 5, 7, 11, 13, 17, and 19. The composite numbers less than or equal to 25 are 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24 and 25.

### Equivalent Fractions

If Ahmed eats $\frac{1}{2}$ of a pizza and Salim eats $\frac{2}{4}$ of the pizza, have they eaten the same amount of pizza? In other words, does $\frac{1}{2} = \frac{2}{4}$?

**Equivalent Fractions**

Equivalent fractions are fractions that have the same value.

For example, $\frac{2}{4}$, $\frac{3}{6}$, and $\frac{4}{8}$ are all equivalent to $\frac{1}{2}$.
In working with equivalent fractions, you saw that there are many ways to write fractions that have the same value. A fraction is considered simplified if there are no common factors, other than 1, in the numerator and denominator.

**Simplified Fraction**

A fraction is considered simplified if there are no common factors in the numerator and denominator.

For example, $\frac{2}{3}$ is simplified, but $\frac{10}{15}$ is not simplified. [5 is a common factor of 10 and 15.]

### Multiplication of fractions

**Fraction Multiplication**

If $a$, $b$, $c$ and $d$ are numbers where $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

**Example 1.15:** Multiply and write the answer in simplified form:

**a.** $\frac{3}{4} \cdot \frac{1}{5}$ **b.** $-\frac{5}{8}\left(-\frac{2}{3}\right)$

**Solution:**

**a.** $\frac{3}{4} \cdot \frac{1}{5} = \frac{3 \cdot 1}{4 \cdot 5} = \frac{3}{20}$ Multiply the numerators; multiply the denominators.

**b.** $-\frac{5}{8}\left(-\frac{2}{3}\right) = \frac{5 \cdot 2}{8 \cdot 3}$ The signs are the same, so the product is positive. Multiply the numerators; multiply the denominators.

$= \frac{5 \cdot 2}{2 \cdot 4 \cdot 3}$ Show common factors and then remove them.

$= \frac{5}{12}$ Multiply remaining factors.

## Reciprocals

**Reciprocal**

The reciprocal of the fraction $\frac{a}{b}$ is $\frac{b}{a}$, where $a \neq 0$ and $b \neq 0$.

The product of a number and its reciprocal is 1.

## Division of Fractions

**Fraction Division**

If $a$, $b$, $c$ and $d$ are numbers where $b \neq 0$, $c \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

**Example 1.16:** Divide, and write the answer in simplified form:

**a.** $\frac{2}{5} \div -\frac{3}{7}$ **b.** $-\frac{3}{4} \div \left(-\frac{7}{8}\right)$

**Solution:**

**a.**

$$\frac{2}{5} \div -\frac{3}{7} = \frac{2}{5}\left(-\frac{7}{3}\right)$$ Multiply the first fraction by the reciprocal of the second.

$$= -\frac{14}{15}$$ Multiply. The product is negative.

**b.**

$$-\frac{3}{4} \div \left(-\frac{7}{8}\right) = -\frac{3}{4} \cdot \left(-\frac{8}{7}\right)$$ Multiply the first fraction by the reciprocal of the second.

$$= \frac{3 \cdot 8}{4 \cdot 7}$$ Multiply. Remember to determine the sign first.

$$= \frac{3 \cdot \cancel{4} \cdot 2}{\cancel{4} \cdot 7}$$ Rewrite to show common factors.

$$= \frac{6}{7}$$ Remove common factors and simplify.

## Add and Subtract Fractions with Common Denominators

**Fraction Addition and Subtraction**

If $a$, $b$, and $c$ are numbers where $c \neq 0$, then

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \quad \text{and} \quad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

**Example 1.17:** Simplify:

**a.** $\frac{3}{5}+\frac{1}{5}$ **b.** $\frac{23}{24}-\frac{14}{24}$

**Solution:**

**a.**

$$\frac{3}{5}+\frac{1}{5} = \frac{3+1}{5}$$ Add the numerators and place the sum over the common denominator.

$$= \frac{4}{5}$$ Simplify.

**b.**

$$\frac{23}{24}-\frac{14}{24} = \frac{23-14}{24}$$ Subtract the numerators and place the difference over the common denominator.

$$= \frac{9}{24}$$ Simplify the numerator.

$$= \frac{3}{8}$$ Simplify the fraction by removing common factors.

### Add and Subtract Fractions with Different Denominators

When we add fractions with different denominators we have to convert them to equivalent fractions with a common denominator. Now lets see what you need to do with fractions that have different denominators.

#### Least Common Denominator

The **least common denominator** (LCD) of two fractions is the **least common multiple** (LCM) of their denominators.

***How To* :: Find the Least Common Denominator (LCD) of Two Fractions.**




Step 1. Factor each denominator into its primes.

Step 2. List the primes, matching primes in columns when possible.

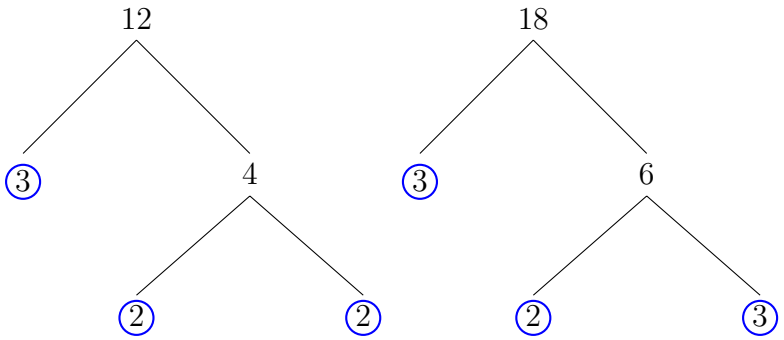
Step 3. Bring down the columns.

Step 4. Multiply the factors. The product is the LCM of the denominators.

Step 5. The LCM of the denominators is the LCD of the fractions.

**Example 1.18:** Find the LCD for the fractions $\frac{7}{12}$ and $\frac{5}{18}$.

**Solution:** Factor each denominator into its primes.




List the primes of 12 and the primes of 18 lining them up in columns when possible.

$12 = 2 \cdot 2 \cdot 3$

$18 = 2 \cdot 3 \cdot 3$

Bring down the columns.

$12 = 2 \cdot 2 \cdot 3$

$18 = 2 \cdot \quad 3 \cdot 3$

$\mathbf{LCD} = 2 \cdot 2 \cdot 3 \cdot 3$

The product is the LCM. The LCM of 12 and 18 is 36, so the LCD of $\frac{7}{12}$ and $\frac{5}{18}$ is 36.

### Add and Subtract Fractions with Different Denominators

Once we have converted two fractions to equivalent forms with common denominators, we can add or subtract them by adding or subtracting the numerators.

***How To* :: Add or Subtract Fractions with Different Denominators.**

How?

Step 1. Find the LCD.
Step 2. Convert each fraction to an equivalent form with the LCD as the denominator.
Step 3. Add or subtract the fractions.
Step 4. Write the result in simplified form.

**Example 1.19:** Add: $\frac{7}{12} + \frac{5}{18}$.

**Solution:** First find LCD of $\frac{7}{12}$ $\frac{5}{18}$.

$$\frac{7}{12} + \frac{5}{18} = \frac{7 \cdot 3}{12 \cdot 3} + \frac{5 \cdot 2}{18 \cdot 2}$$ Change into equivalent fractions with the LCD 36.

$$= \frac{21}{36} + \frac{10}{36} = \frac{31}{36}$$ Simplify the numerators and denominators.

$12 = 2 \cdot 2 \cdot 3$

$18 = 2 \cdot \quad 3 \cdot 3$

$\mathbf{LCD} = 2 \cdot 2 \cdot 3 \cdot 3$

$\mathbf{LCD} = 36$

### 1.3.2 Decimals

#### Add and Subtract Decimals

***How To* :: Add and Subtract Decimals.**




Step 1. Write the numbers vertically so the decimal points line up.
Step 2. Use zeros as place holders, as needed.
Step 3. Add or subtract the numbers as if they were whole numbers. Then place the decimal in the answer under the decimal points in the given numbers.

**Example 1.20:** **a. Add:** 5.7 + 11.9 **b. Add:** 23.5 + 41.38

**Solution:**

**a.** Write the numbers vertically so the decimal points line up

$$\begin{array}{r} 5.7 \\ +\ 11.9 \\ \hline 17.6 \end{array}$$

**b.** Write the numbers vertically so the decimal points line up and Place 0 as a place holder after the 5 in 23.5, so that both numbers have two decimal places

$$\begin{array}{r} 23.50 \\ +\ 41.38 \\ \hline 64.88 \end{array}$$

**Example 1.21:** **a. Subtract:** 20−14.65 **b. Subtract:** 2.51−7.4

**Solution:**

**a.** Write the numbers vertically so the decimal points line up. Remember 20 is a whole number, so place the decimal point after the 0.

$$\begin{array}{r} 20.00 \\ -\ 14.65 \\ \hline 5.35 \end{array}$$

**b.** If we subtract 7.4 from 2.51, the answer will be negative since $7.4 > 2.51$. To subtract easily, we can subtract 2.51 from 7.4. Then we will place the negative sign in the result.

Write the numbers vertically so the decimal points line up. Place 0 as a place holder after the 4 in 7.4, so that both numbers have two decimal places.

$$\begin{array}{r} 7.40 \\ -\ 2.51 \\ \hline 4.89 \end{array}$$

Remember that we are really subtracting 2.51−7.4 so the answer is negative.

2.51−7.4 = −4.89

### 1.3.3 Ratio and Percent

#### Ratio

**Ratios**

A ratio compares two numbers or two quantities that are measured with the same unit. The ratio of $a$ to $b$ is written $a$ to $b$, $\frac{a}{b}$, or a:b.

Because a ratio compares two quantities, we would leave a ratio as $\frac{4}{1}$ instead of simplifying it to 4 so that we can see the two parts of the ratio.

**Example 1.22:** Write each ratio as a fraction:

**a.** 15 to 27 **b.** 45 : 18

**Solution:**

**a.** 15 to 27 $\frac{15}{27} = \frac{3 \cdot 5}{3 \cdot 9} = \frac{5}{9}$ Write as a fraction with the first number in the numerator and the second in the denominator.

**b.** 45 : 18 $\frac{45}{18} = \frac{9 \cdot 5}{9 \cdot 2} = \frac{5}{2}$

#### Percent

**Percent**

A percent is a ratio whose denominator is 100. We use the percent symbol %, to show percent.

**Example 1.23:** According to a survey, 89 % of college students have a smartphone. Write this percent as a ratio.

**Solution:** $89\% = \frac{89}{100}$

***How To* :: Convert a Percent to a Fraction**

How?

Step 1. Write the percent as a ratio with the denominator 100.

Step 2. Simplify the fraction if possible.

**Example 1.24:** Convert each percent to a fraction:

**a.** 36% **b.** 125%

**Solution:**

**a.** $36\% = \frac{36}{100} = \frac{9 \cdot 4}{25 \cdot 4} = \frac{9}{25}$ Write as a ratio with denominator 100 and simplify.

**b.** $125\% = \frac{125}{100} = \frac{5 \cdot 25}{4 \cdot 25} = \frac{5}{4}$ Write as a ratio with denominator 100 and simplify.

In Decimals, we learned how to convert fractions to decimals. To convert a percent to a decimal, we first convert it to a fraction and then change the fraction to a decimal.

***How To* :: Convert a Pecent to a Decimal**




Step 1. Write the percent as a ratio with the denominator 100.

Step 2. Convert the fraction to a decimal by dividing the numerator by the denominator.

**Example 1.25:** Convert each percent to a decimal:

**a.** 135% **b.** 12.5% **c.** 7 %

**Solution:** Because we want to change to a decimal, we will leave the fractions with denominator 100 instead of removing common factors.

**a.** $135\% = \frac{135}{100} = 1.35$

**b.** $12.5\% = \frac{12.5}{100} = 0.125$

**c.** $7\% = \frac{7}{100} = 0.07$

*Do you see the pattern?*

To convert a percent number to a decimal number, we move the decimal point two places to the left and remove the % sign. (Sometimes the decimal point does not appear in the percent number, but just like we can think of the integer 6 as 6.0, we can think of 6% as 6.0%. ) Notice that we may need to add zeros in front of the number when moving the decimal to the left.

**Example 1.26:** Convert 77% to: **(a)** a fraction **(b)** a decimal.

**Solution:**

**a.** $77\% = \frac{77}{100}$ Write as a ratio with denominator 100.

**b.** $77\% = \frac{77}{100} = 0.77$ Write as a ratio with denominator 100 and change the fraction to a decimal

### Convert Decimals and Fractions to Percents

To convert a decimal to a percent, remember that percent means per hundred. If we change the decimal to a fraction whose denominator is 100, it is easy to change that fraction to a percent.

***How To* :: Convert a Decimal to a Percent**




Step 1. Write the decimal as a fraction.

Step 2. If the denominator of the fraction is not 100, rewrite it as an equivalent fraction with denominator 100.

Step 3. Write this ratio as a percent.

**Example 1.27:** Convert each decimal to a percent: **(a.)** 0.05 **(b.)** 0.83

**Solution:**

**a.** $0.05 = \frac{5}{100}$ Write as a ratio with denominator 100.

$= 5\%$ Write this ratio as a percent.

**b.** $0.83 = \frac{83}{100}$ Write as a ratio with denominator 100.

$= 83\%$ Write this ratio as a percent.

Do you see the pattern? To convert a decimal to a percent, we move the decimal point two places to the right and then add the percent sign.

**Example 1.28:** Convert each fraction or mixed number to a percent:

**a.** $\frac{3}{4}$ **b.** $\frac{11}{8}$ **c.** $2\frac{1}{5}$

**Solution:**

**a.** $\frac{3}{4} = \frac{3}{4} \times 100 = \frac{3 \times 100}{4} = \frac{3 \times 4 \times 25}{4} = 75\%$

**b.** $\frac{11}{8} = \frac{11}{8} \times 100 = \frac{11 \times 100}{8} = \frac{11 \times 4 \times 25}{2 \times 4} = 137.5\%$

**c.** $2\frac{1}{5} = \frac{11}{5} \times 100 = \frac{11 \times 100}{5} = \frac{11 \times 5 \times 20}{5} = 220\%$

## 1.3 Section Exercises

*In the following exercises, rewrite the improper fraction as a mixed number.*

**1.** $\frac{3}{2}$ **2.** $\frac{5}{3}$ **3.** $\frac{11}{4}$

*In the following exercises, rewrite the mixed number as an improper fraction.*

**4.** $1\frac{2}{3}$ **5.** $2\frac{2}{5}$ **6.** $2\frac{1}{4}$

*In the following exercises, simplify each fraction. Do not convert any improper fractions to mixed numbers.*

**7.** $\frac{7}{21}$ **8.** $\frac{8}{24}$ **9.** $\frac{20}{15}$

*In the following exercises, multiply, and write the answer in simplified form.*

**10.** $\frac{2}{5} \cdot \frac{1}{3}$ **11.** $\frac{1}{2} \cdot \frac{3}{8}$ **12.** $-\frac{5}{9} \cdot \frac{3}{10}$

**13.** $-\frac{3}{8} \cdot \frac{4}{15}$ **14.** $\frac{7}{12}\left(-\frac{8}{21}\right)$ **15.** $\left(-\frac{9}{10}\right)\left(\frac{25}{33}\right)$

*In the following exercises, divide, and write the answer in simplified form.*

**16.** $\frac{1}{2} \div \frac{1}{4}$ **17.** $\frac{3}{4} \div \frac{2}{3}$ **18.** $-\frac{4}{5} \div \frac{4}{7}$

**19.** $-\frac{7}{9} \div \left(-\frac{7}{9}\right)$ **20.** $\frac{3}{4} \div \frac{x}{11}$ **21.** $\frac{7p}{12} \div \frac{21p}{8}$

*In the following exercises, find each sum or difference.*

**22.** $\frac{4}{9} + \frac{1}{9}$ **23.** $\frac{6}{13} + \left(-\frac{10}{13}\right) + \left(-\frac{12}{13}\right)$ **24.** $\frac{x}{4} + \frac{3}{4}$

**25.** $\frac{4}{5} - \frac{1}{5}$ **26.** $\frac{11}{15} - \frac{7}{15}$ **27.** $\frac{7}{12} - \frac{2}{12}$

*In the following exercises, add or subtract. Write the result in simplified form.*

**28.** $\frac{1}{3} + \frac{1}{5}$ **29.** $\frac{1}{4} + \frac{1}{10}$ **30.** $\frac{7}{12} + \frac{5}{8}$

**31.** $\frac{5}{12} + \frac{3}{18}$ **32.** $\frac{7}{12} - \frac{9}{16}$ **33.** $\frac{11}{12} - \frac{3}{8}$

*In the following exercises, write each ratio as a fraction.*

**34.** 20 to 36 **35.** 45 to 54 **36.** 42 to 48

*In the following exercises, write each percent as a ratio.*

**37.** 12%. **38.** 35% **39.** 2.5%

*In the following exercises, convert each percent to a fraction and simplify all fractions.*

**40.** 4% **41.** 120% **42.** 12.5%

*In the following exercises, convert each percent to a decimal.*

**43.** 250% **44.** 9% **45.** 15%

**46.** 39.3% **47.** 7.5% **48.** 100%

*In the following exercises, convert each decimal to a percent.*

**49.** 0.01 **50.** 0.18 **51.** 1.35

*In the following exercises, convert each fraction to a percent.*

**52.** $\frac{1}{4}$ **53.** $\frac{1}{5}$ **54.** $5\frac{1}{4}$

# Chapter 2

# EXPONENTS AND RADICALS

## Contents


**2.1 Integer Exponents**

2.1.1 Rules of Integer Exponents

**2.2 Rational Exponents**

2.2.1 Simplifying Expressions with Rational Exponents

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**2.3 Radicals**

2.3.1 Converting expressions between radicals and exponents

2.3.2 Properties of Radicals

2.3.3 Add, Subtract, and Multiply Radical Expressions

2.3.4 Rationalizing the Denominator

2.3.5 Solve Radical Equations


**Learning outcome covered:**

**c.** Demonstrate an understanding of the exponent laws, and apply them to simplify expression and manipulate fractions, ratios, decimals, and percentages.[partial]

**l.** Simplify rational expressions and rationalize numerators or denominators.

**m.** Solve linear equations, equations involving radicals, fractional expression and inequalities.

**Learning Objectives**

**At the end of this module, the students will be able to:**

- Use the product rule of exponents
- Use the quotient rule of exponents

- Use the power rule of exponents
- Use the zero exponent rule
- Use the rule of negative exponents
- Find the power of a product and a quotient
- Simplify Exponential and Radical expressions
- Add and Subtract Radical Expressions
- Multiply and Divide Radical Expressions
- Rationalize denominator
- Solve Radical Equations

## Introduction

Using a calculator, we enter $2048 \times 1536 \times 48 \times 24 \times 3600$ and press ENTER. The calculator displays 1.304596316E13. What does this mean? The E13 portion of the result represents the exponent 13 of ten, so there are a maximum of approximately $1.3 \times 10^{13}$ bits of data in that one-hour film. In this chapter, we study about integer and rational exponents and review the properties of exponents. Then we define radicals from exponents and explain how to solve radical equations.

## 2.1 Integer Exponents

**Definition of Integer Exponents**

When we multiply a number by itself, we **square** it or **raise** it to a power of 2. For example, $4^2 = 4 \cdot 4 = 16$. We can raise any number to any power. In general, the exponential notation $a^n$ means that the number or variable $a$ is used as a factor $n$ times.

$$\overbrace{a \cdot a \cdot a \cdots \cdot a}^{n \text{ times}}$$

In this notation, $a^n$ is read as the $n^{th}$ power of $a$, where $a$ is called the **base** and $n$ is called the **exponent**.

### 2.1.1 Rules of Integer Exponents

***1. Product Rule of Exponents***

Consider the product $x^3 \cdot x^4$. Both terms have the same base $x$, but they are raised to different exponents.

$$\begin{aligned} x^3 \cdot x^4 &= \overbrace{x \cdot x \cdot x}^{3 \text{ factors}} \cdot \overbrace{x \cdot x \cdot x \cdot x}^{4 \text{ factors}} \\ &= \overbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}^{7 \text{ factors}} \\ &= x^7 \end{aligned}$$

The result is that $x^3 \cdot x^4 = x^7$.

Notice that the exponent of the product is the sum of the exponents of the terms. In other words, when multiplying exponential expressions with the same base, we write the result with the common base and add the exponents.

***The product rule of exponents***

For any real number $a$ and natural numbers $m$ and $n$, the product rule of exponents states that

$$a^m \cdot a^n = a^{m+n}$$

**Example 2.1:** Write each of the following products with a single base. Do not simplify further.

**a.** $t^5 \cdot t^3$ **b.** $(-3)^5 \cdot (-3)$ **c.** $x^2 \cdot x^5 \cdot x^3$

**Solution:**

**a.** $t^5 \cdot t^3 = t^{5+3} = t^8$ [Use the product rule to simplify]

**b.** $(-3)^5 \cdot (-3) = (-3)^{5+1} = (-3)^6$ [Use the product rule to simplify]

**c.** $x^2 \cdot x^5 \cdot x^3 = (x^2 \cdot x^5) \cdot x^3 = (x^{2+5}) \cdot x^3 = x^7 \cdot x^3 = x^{7+3} = x^{10}$
Notice we get the same result by adding the three exponents in one step.
$x^2 \cdot x^5 \cdot x^3 = x^{2+5+3} = x^{10}$

### *2. Quotient Rule of Exponents*

The quotient rule of exponents allows us to simplify an expression that divides two numbers with the same base but different exponents.

***The quotient rule of exponents***

For any real number $a \neq 0$ and positive integers $m$ and $n$, the power rule of exponents states that

$$\frac{a^m}{a^n} = a^{m-n}$$

**Example 2.2:** Write each of the following expressions with a single base. Do not simplify further.

**a.** $\dfrac{(-2)^{14}}{(-2)^9}$ **b.** $\dfrac{t^{23}}{t^{15}}$

**Solution:**

Use the quotient rule to simplify each expression.

**a.** $\dfrac{(-2)^{14}}{(-2)^9} = (-2)^{14-9} = (-2)^5$ **b.** $\dfrac{t^{23}}{t^{15}} = t^{23-15} = t^8$

### 3. Power Rule of Exponents

Be careful to distinguish between uses of the product rule and the power rule. When using the product rule, different terms with the same bases are raised to exponents. In this case, you add the exponents. When using the power rule, a term in exponential notation is raised to a power. In this case, you multiply the exponents.

**The power rule of exponents**

For any real number $a$ and positive integers $m$ and $n$, the power rule of exponents states that

$$(a^m)^n = a^{mn}$$

**Example 2.3:** Write each of the following expressions with a single base. Do not simplify further.

**a.** $((x)^2)^7$ **b.** $((2t)^5)^3$

**Solution:**

Use the quotient rule to simplify each expression.

**a.** $((x)^2)^7 = x^{2\cdot 7} = x^{14}$ **b.** $((2t)^5)^3 = (2t)^{5\cdot 3} = (2t)^{15}$

### 4. Zero Exponent Rule of Exponents

Return to the quotient rule. What would happen if $m = n$? In this case, we would use the zero exponent rule to simplify the expression to 1. To see how this is done, let us begin with an example.

$$\frac{t^8}{t^8} = \frac{\cancel{t^8}}{\cancel{t^8}} = 1$$

If we were to simplify the original expression using the quotient rule, we would have

$$\frac{t^8}{t^8} = t^{8-8} = t^0$$

If we equate the two answers, the result is $t^0 = 1$. This is true for any non-zero real number, or any variable representing a real number. The sole exception is the expression $0^0$.

**The zero exponent rule**

For any non-zero real number $a$, the zero exponent rule states that

$$a^0 = 1$$

**Example 2.4:** Simplify each expression using the zero exponent rule.

**a.** $\dfrac{c^3}{c^3}$ **b.** $\dfrac{-3x^5}{x^5}$

**Solution:**

**a.**
$$\begin{aligned} \frac{c^3}{c^3} &= c^{3-3} && \text{Quotient rule} \\ &= c^0 && \text{Simplify} \\ &= 1 && \text{Zero exponent} \end{aligned}$$

**b.**
$$\begin{aligned} \frac{-3x^5}{x^5} &= -3 \cdot \frac{x^5}{x^5} \\ &= -3 \cdot x^{5-5} && \text{Quotient rule} \\ &= -3 \cdot x^0 && \text{Simplify} \\ &= -3 \cdot 1 && \text{Zero exponent} \\ &= -3 \end{aligned}$$

### *5. Rule of Negative Exponents*

When $m < n$ that is, where the difference $m-n$ is negative$(-)$ we can use the rule of negative exponents to simplify the expression to its reciprocal.

Divide one exponential expression by another with a larger exponent. Use our example, $\frac{h^3}{h^5}$.

$$\frac{h^3}{h^5} = \frac{h \cdot h \cdot h}{h \cdot h \cdot h \cdot h \cdot h} = \frac{\cancel{h} \cdot \cancel{h} \cdot \cancel{h}}{\cancel{h} \cdot \cancel{h} \cdot \cancel{h} \cdot h \cdot h} = \frac{1}{h \cdot h} = \frac{1}{h^2}$$

If we were to simplify the original expression using the quotient rule, we would have

$$\frac{h^3}{h^5} = h^{3-5} = h^{-2}$$

Putting the answers together, we have $h^{-2} = \frac{1}{h^2}$. This is true for any non-zero real number, or any variable representing a non-zero real number.

A factor with a negative exponent becomes the same factor with a positive exponent if it is moved across the fraction barfrom numerator to denominator or vice versa.

$$a^{-n} = \frac{1}{a^n} \text{ and } a^n = \frac{1}{a^{-n}}$$

***The rule of negative exponents***

For any non-zero real number $a$ and natural number $n$, the rule of negative exponents states that $a^{-n} = \frac{1}{a^n}$

**Example 2.5:** Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

**a.** $\frac{\theta^3}{\theta^{10}}$ **b.** $\frac{z^2 \cdot z}{z^4}$ **c.** $\frac{(-5t^3)^4}{(-5t^3)^8}$

**Solution:**

**a.** $\frac{\theta^3}{\theta^{10}} = \theta^{3-10} = \theta^{-7} = \frac{1}{\theta^7}$

**b.** $\frac{z^2 \cdot z}{z^4} = \frac{z^3}{z^4} = z^{3-4} = z^{-1} = \frac{1}{z}$

**c.** $\frac{(-5t^3)^4}{(-5t^3)^8} = (-5t^3)^{4-8} = (-5t^3)^{-4} = \frac{1}{(-5t^3)^4}$

### 6. Power of a Product

To simplify the power of a product of two exponential expressions, we can use the power of a product rule, which breaks up the power of a product of factors into the product of the powers of the factors.

***The power of a product rule of exponents***

For any non-zero real number $a$ and natural number $n$, the power of a product rule states that

$$(ab)^n = a^n \cdot b^n$$

**Example 2.6:** Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

**a.** $(ab^2)^3$ **b.** $(2t)^{15}$ **c.** $(-2w^3)^3$ **d.** $\dfrac{1}{(-2z)^4}$ **e.** $(e^{-2}f^2)^7$

**Solution:** Use the product and quotient rules and the new definitions to simplify each expression.

**a.** $(ab^2)^3 = a^3 \cdot (b^2)^3 = a^{1\cdot 3} \cdot b^{2\cdot 3} = a^3b^6$

**b.** $(2t)^{15} = (2)^{15} \cdot (t)^{15} = (2)^{15}(t)^{15}$

**c.** $(-2w^3)^3 = (-2)^3 \cdot (w^3)^3 = -8 \cdot w^{3\cdot 3} = -8w^9$

**d.** $\dfrac{1}{(-2z)^4} = \dfrac{1}{(-2)^4 \cdot (z)^4} = \dfrac{1}{16z^4}$

**e.** $(e^{-2}f^2)^7 = (e^{-2})^7 \cdot (f^2)^7 = e^{-2\cdot 7} \cdot f^{2\cdot 7} = e^{-14}f^{14} = \dfrac{f^{14}}{e^{14}}$

---

### 7. Power of a Quotient

To simplify the power of a quotient of two expressions, we can use the power of a quotient rule, which states that the power of a quotient of factors is the quotient of the powers of the factors.

***The power of a quotient rule of exponents***

For any real numbers $a$ and $b$ and any integer $n \neq 0$, the power of a quotient rule of exponents states that

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

**Example 2.7:** Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

**a.** $\left(\dfrac{4}{z^{11}}\right)^3$ **b.** $\left(\dfrac{p}{q^3}\right)^6$ **c.** $\left(\dfrac{-1}{t^2}\right)^{27}$

**Solution:**

**a.** $\left(\dfrac{4}{z^{11}}\right)^3 = \dfrac{4^3}{(z^{11})^3} = \dfrac{64}{z^{11\cdot 3}} = \dfrac{64}{z^{33}}$

**b.** $\left(\dfrac{p}{q^3}\right)^6 = \dfrac{p^6}{(q^3)^6} = \dfrac{p^6}{q^{3\cdot 6}} = \dfrac{p^6}{q^{18}}$

**c.** $\left(\dfrac{-1}{t^2}\right)^{27} = \dfrac{(-1)^{27}}{(t^2)^{27}} = \dfrac{-1}{t^{2\cdot 27}} = \dfrac{-1}{t^{54}}$

---

### Simplifying Exponential Expressions

Recall that to simplify an expression means to rewrite it by combining terms or exponents; in other words, to write the expression more simply with fewer terms. The rules of exponents may be combined to simplify expressions.

**Example 2.8:** Simplify each expression and write the answer with positive exponents only.

**a.** $(6m^2n^{-1})^3$ **b.** $6^5 \cdot 6^{-4} \cdot 6^{-3}$ **c.** $\left(\dfrac{u^{-1}v}{v^{-1}}\right)^2$

**d.** $(-2a^3b^{-1})(5a^{-2}b^2)$ **e.** $(x^2y)^4(x^2y)^{-4}$ **f.** $\dfrac{(2w^2)^4}{(6w^{-2})^2}$

**Solution:**

**a.** $(6m^2n^{-1})^3$

$= 6^3(m^2)^3(n^{-1})^3$ The power of a product rule

$= 6^3m^{2\cdot 3}n^{-1\cdot 3}$ The power rule

$= 216m^6n^{-3}$ Simplify

$= \dfrac{216m^6}{n^3}$ The negative exponent rule

**b.** $6^5 \cdot 6^{-4} \cdot 6^{-3}$

$= 6^{5-4-3}$ The product rule

$= 6^{-2}$ Simplify

$= \dfrac{1}{6^2}$ or $\dfrac{1}{36}$ The negative exponent rule

**c.** $\left(\dfrac{u^{-1}v}{v^{-1}}\right)^2$

$= \dfrac{(u^{-1}v)^2}{(v^{-1})^2}$ The power of a quotient rule

$= \dfrac{u^{-2}v^2}{v^{-2}}$ The power of a product rule

$= u^{-2}v^{2-(-2)}$ The quotient rule

$= u^{-2}v^4$ Simplify.

$= \dfrac{v^4}{u^2}$ The negative exponent rule

| | | | |
|---|---|---|---|
| **d.** | $(-2a^3b^{-1})(5a^{-2}b^2)$ | $= -2 \cdot 5 \cdot a^3 \cdot a^{-2} \cdot b^{-1} \cdot b^2$ | Commutative and associative laws of multiplication |
| | | $= -10 \cdot a^{3-2} \cdot b^{-1+2}$ | The product rule |
| | | $= -10ab$ | Simplify. |
| **e.** | $(x^2y)^4(x^2y)^{-4}$ | $= (x^2y)^{4-4}$ | The product rule |
| | | $= (x^2y)^0$ | Simplify. |
| | | $= 1$ | The zero exponent rule |
| **f.** | $\dfrac{(2w^2)^4}{(6w^{-2})^2}$ | $= \dfrac{2^4(w^2)^4}{6^2(w^{-2})^2}$ | The power of a product rule |
| | | $= \dfrac{2^4w^{2\cdot4}}{6^2w^{-2\cdot2}}$ | The power rule |
| | | $= \dfrac{16w^8}{36w^{-4}}$ | Simplify. |
| | | $= \dfrac{4w^{8-(-4)}}{9}$ | The quotient rule and reduce fraction |
| | | $= \dfrac{4w^{12}}{9}$ | Simplify. |

## 2.1 Section Exercises

### Numeric

*For the following exercises, simplify the given expression. Write answers with positive exponents.*

**1.** $9^2$ **2.** $15^{-2}$ **3.** $3^2 \cdot 3^3$ **4.** $4^4 \div 4$

**5.** $(2^2)^{-2}$ **6.** $(5-8)^0$ **7.** $11^3 \div 11^4$ **8.** $6^5 \cdot 6^{-7}$

*For the following exercises, write each expression with a single base. Do not simplify further. Write answers with positive exponents.*

**9.** $4^2 \cdot 4^3 \div 4^{-4}$ **10.** $\dfrac{6^{12}}{6^9}$ **11.** $(12^3 \cdot 12)^{10}$ **12.** $10^6 \div (10^{10})^{-2}$

**13.** $7^6 \cdot 7^{-3}$ **14.** $(3^3 \div 3^4)^5$

### Algebraic

*For the following exercises, simplify the given expression. Write answers with positive exponents.*

**15.** $\dfrac{a^3a^2}{a}$ **16.** $\dfrac{mn^2}{m^{-2}}$ **17.** $(b^3c^4)^2$ **18.** $\left(\dfrac{x^{-3}}{y^2}\right)^{-5}$

**19.** $(ab^2) \div d^{-3}$ **20.** $(w^0x^5)^{-1}$ **21.** $\dfrac{m^4}{n^0}$ **22.** $y^{-4}(y^2)^2$

**23.** $\dfrac{p^{-4}q^2}{p^2q^{-3}}$ **24.** $(l \times w)^2$ **25.** $(y^7)^3 \div x^{14}$ **26.** $\left(\dfrac{a}{2^3}\right)^2$

**27.** $5^2m \div 5^0m$ **28.** $\dfrac{(16x^2}{y^{-1}}$ **29.** $\dfrac{2^3}{(3a)^{-2}}$ **30.** $(ma^6)^2\dfrac{1}{m^3a^2}$

**31.** $(b^{-3}c)^3$ **32.** $(x^2y^{13} \div y^0)^2$ **33.** $(9z^3)^{-2}y$

**Extensions**

*For the following exercises, simplify the given expression. Write answers with positive exponents.*

**34.** $\left(\dfrac{3^2}{a^3}\right)^{-2}\left(\dfrac{a^4}{2^2}\right)^2$ **35.** $(6^2-24)^2 \div \left(\dfrac{x}{y}\right)^{-5}$ **36.** $\dfrac{m^2n^3}{a^2c^{-3}} \cdot \dfrac{a^{-7}n^{-2}}{m^2c^4}$

**37.** $\left(\dfrac{x^6y^3}{x^3y^{-3}} \cdot \dfrac{y^{-7}}{x^{-3}}\right)^{10}$ **38.** $\left(\dfrac{(ab^2c)^{-3}}{b^{-3}}\right)^2$

## 2.2 Rational Exponents

Rational exponents are exponents that are fractions, where the numerator is a power and the denominator is a root. For example, $16^{\frac{1}{2}}$ is another way of writing $\sqrt{16}$, $8^{\frac{1}{3}}$ is another way of writing $\sqrt[3]{8}$ . The ability to work with rational exponents is a useful skill, as it is highly applicable in calculus.

***Definition***

A rational exponent indicates a power in the numerator and a root in the denominator. There are multiple ways of writing an expression, a variable, or a number with a rational exponent:

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (a^m)^{\frac{1}{n}}$$

**Example 2.9:** **Simplify:** **a.** $125^{\frac{2}{3}}$ **b.** $-25^{-\frac{3}{2}}$ **c.** $32^{-\frac{2}{5}}$

**Solution:**

**a.** $125^{\frac{2}{3}} = (5^3)^{\frac{2}{3}} = 5^{3\cdot\frac{2}{3}} = 5^2 = 25$

**b.** $-25^{-\frac{3}{2}} = -(5^2)^{-\frac{3}{2}} = -5^{-3} = -\frac{1}{5^3} = -\frac{1}{125}$

**c.** $32^{-\frac{2}{5}} = (2^5)^{-\frac{2}{5}} = 2^{5\cdot-\frac{2}{5}} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

**Properties of Rational Exponents** We will list the Properties of Exponenets here to have them for reference as we simplify expressions.

***Rational exponents***

If $a$ and $b$ are real numbers and $m$ and $n$ are rational numbers, then

| | |
|---|---|
| **Product Property** | $a^m \cdot a^n = a^{m+n}$ |
| **Quotient Property** | $\frac{a^m}{a^n} = a^{m-n}$ |
| **Power Property** | $(a^m)^n = a^{mn}$ |
| **Zero Exponent Property** | $a^0 = 1, \quad a \neq 0$ |
| **Negative Exponent Property** | $a^{-n} = \frac{1}{a^n}, \quad a \neq 0$ |
| **Product to a Power** | $(ab)^m = a^m b^m$ |
| **Quotient to a Power** | $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, \quad b \neq 0$ |

### 2.2.1 Simplifying Expressions with Rational Exponents

**Example 2.10: Simplify:**

**a.** $x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$ **b.** $(z^9)^{\frac{2}{3}}$ **c.** $\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$

**Solution:**

**a.** $x^{\frac{1}{2}} \cdot x^{\frac{5}{6}} = x^{\frac{1}{2}+\frac{5}{6}} = x^{\frac{8}{6}} = x^{\frac{4}{3}}$

**b.** $(z^9)^{\frac{2}{3}} = z^{9\cdot\frac{2}{3}} = z^{\cancel{9}^{3}\cdot\frac{2}{\cancel{3}}} = z^6$

**c.** $\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}} = x^{\frac{1}{3}-\frac{5}{3}} = x^{-\frac{4}{3}} = \frac{1}{x^{\frac{4}{3}}}$

**Example 2.11: Simplify:**

**a.** $\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$ **b.** $\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$

**Solution:**

**a.** $\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$

$= 27^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}} = (3^3)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$ The power of a product rule & write 27 as a power of 3.

$= 3^{3\cdot\frac{2}{3}}u^{\frac{1}{2}\cdot\frac{2}{3}}$ The power rule

$= 3^2u^{\frac{1}{3}} = 9u^{\frac{1}{3}}$ Simplify

**b.** $\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$

$= \left(m^{\frac{2}{3}}\right)^{\frac{3}{2}}\left(n^{\frac{1}{2}}\right)^{\frac{3}{2}}$ The power of a quotient rule

$= m^{\frac{2}{3}\cdot\frac{3}{2}}n^{\frac{1}{2}\cdot\frac{3}{2}}$ The power rule

$= mn^{\frac{3}{4}}$ Simplify

We will use both the Product Property and the Quotient Property in the next example.

**Example 2.12:** **Simplify:** **a.** $\dfrac{x^{\frac{3}{4}}\cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$ **b.** $\left(\dfrac{16x^{\frac{4}{3}}y^{-\frac{5}{6}}}{x^{-\frac{2}{3}}y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$

**Solution:**

**a.** $\dfrac{x^{\frac{3}{4}}\cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$

$= \dfrac{x^{\frac{3}{4}-\frac{1}{4}}}{x^{-\frac{6}{4}}}$ The product rule

$= \dfrac{x^{\frac{2}{4}}}{x^{-\frac{6}{4}}}$ Simplify

$= x^{\frac{2}{4}+\frac{6}{4}}$ The quotient rule

$= x^{\frac{8}{4}} = x^2$ Simplify

**b.** $\left(\dfrac{16x^{\frac{4}{3}}y^{-\frac{5}{6}}}{x^{-\frac{2}{3}}y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$

$= \left(16x^{\frac{4}{3}+\frac{2}{3}}y^{-\frac{5}{6}-\frac{1}{6}}\right)^{\frac{1}{2}}$ The quotient rule

$= \left(4^2x^{\frac{6}{3}}y^{-\frac{6}{6}}\right)^{\frac{1}{2}} = \left(4^2x^2y^{-1}\right)^{\frac{1}{2}}$ Simplify

$= \left(4^2\right)^{\frac{1}{2}}\left(x^2\right)^{\frac{1}{2}}\left(y^{-1}\right)^{\frac{1}{2}}$ The power of a product rule

$= 4^{2\cdot\frac{1}{2}}x^{2\cdot\frac{1}{2}}y^{-1\cdot\frac{1}{2}}$ The power rule

$= 4xy^{-\frac{1}{2}}$ Simplify

$= \dfrac{4x}{y^{\frac{1}{2}}}$ The negative exponent rule

### 2.2.2 Equations with rational exponents

Sometimes an equation will contain rational exponents. We raise each side of the equation to the power equal to the denominator of the rational exponent.

**Example 2.13: Solve:** $(3x-2)^{\frac{1}{4}}+3=5$

**Solution:**

| | | |
|---|---|---|
| $(3x-2)^{\frac{1}{4}}+3$ | $=5$ | |
| $(3x-2)^{\frac{1}{4}}$ | $=2$ | To isolate the term with the rational exponent, subtract 3 from both sides. |
| $\left[(3x-2)^{\frac{1}{4}}\right]^4$ | $=2^4$ | Raise each side of the equation to the fourth power. |
| $3x-2$ | $=16$ | Simplify. |
| $3x$ | $=18$ | Solve the equation. |
| $x$ | $=6$ | |

## 2.2 Section Exercises

**Use the Laws of Exponents to Simplify Expressions with Rational Exponents**

*In the following exercises, simplify. Assume all variables are positive.*

**1.** **a.** $c^{\frac{1}{4}}\cdot c^{\frac{5}{8}}$ **b.** $6^{\frac{5}{2}}\cdot 6^{\frac{1}{2}}$ **c.** $y^{\frac{1}{2}}\cdot y^{\frac{3}{4}}$ **d.** $q^{\frac{2}{3}}\cdot q^{\frac{5}{6}}$

**2.** **a.** $(p^{12})^{\frac{3}{4}}$ **b.** $(b^{15})^{\frac{3}{5}}$ **c.** $(x^{12})^{\frac{2}{3}}$ **d.** $(h^{6})^{\frac{4}{3}}$

**3.** **a.** $\dfrac{r^{\frac{4}{5}}}{r^{\frac{9}{5}}}$ **b.** $\dfrac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$ **c.** $\dfrac{m^{\frac{5}{8}}}{m^{\frac{13}{8}}}$ **d.** $\dfrac{n^{\frac{3}{5}}}{n^{\frac{8}{5}}}$

**4.** **a.** $\left(27q^{\frac{3}{2}}\right)^{\frac{4}{3}}$ **b.** $\left(m^{\frac{4}{3}}m^{\frac{1}{2}}\right)^{\frac{3}{4}}$ **c.** $\left(4p^{\frac{1}{3}}q^{\frac{1}{2}}\right)^{\frac{3}{2}}$ **d.** $\left(9x^{\frac{2}{5}}y^{\frac{3}{5}}\right)^{\frac{5}{2}}$

**5.** **a.** $\dfrac{r^{\frac{5}{2}}\cdot r^{-\frac{1}{2}}}{r^{-\frac{3}{2}}}$ **b.** $\dfrac{a^{\frac{3}{4}}\cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$ **c.** $\dfrac{c^{\frac{5}{3}}\cdot c^{-\frac{1}{3}}}{c^{-\frac{2}{3}}}$ **d.** $\dfrac{m^{\frac{7}{4}}\cdot m^{-\frac{5}{4}}}{m^{-\frac{2}{4}}}$

**6.** **a** $\left(\dfrac{36s^{\frac{1}{5}}t^{-\frac{3}{2}}}{s^{-\frac{9}{5}}t^{\frac{1}{2}}}\right)^{\frac{1}{2}}$ **b** $\left(\dfrac{27b^{\frac{2}{3}}c^{-\frac{5}{2}}}{b^{-\frac{7}{3}}c^{\frac{1}{2}}}\right)^{\frac{1}{3}}$ **c** $\left(\dfrac{8x^{\frac{5}{3}}y^{-\frac{1}{2}}}{27x^{-\frac{4}{3}}y^{\frac{5}{2}}}\right)^{\frac{1}{3}}$ **d** $\left(\dfrac{16m^{\frac{1}{5}}n^{\frac{3}{2}}}{81m^{\frac{9}{5}}n^{-\frac{1}{2}}}\right)^{\frac{1}{2}}$

*In the following exercises, simplify.*

**7.** **a.** $81^{\frac{1}{2}}$ **b.** $125^{-\frac{1}{3}}$ **c.** $64^{\frac{1}{2}}$ **d.** $64^{\frac{1}{3}}$ **e.** $32^{\frac{1}{5}}$

**8.** **a.** $(-32)^{\frac{1}{5}}$ **b.** $(-8)^{\frac{1}{3}}$ **c.** $-49^{\frac{1}{2}}$ **d.** $49^{-\frac{1}{2}}$ **e.** $-16^{\frac{1}{4}}$

*In the following exercises, simplify.*

**9.** **a.** $25^{\frac{3}{2}}$ **b.** $(-27)^{\frac{2}{3}}$ **c.** $100^{\frac{3}{2}}$ **d.** $81^{-\frac{3}{2}}$ **e.** $27^{-\frac{2}{3}}$

**Solve Equations with Rational Exponents**

*In the following exercises, solve.*

**10.** **a.** $(6x+1)^{\frac{1}{2}}-3=4$ **b.** $(12x-5)^{\frac{1}{3}}+8=3$ **c.** $(5x-4)^{\frac{1}{4}}+7=9$

**11.** **a.** $(3x-5)^{\frac{3}{2}}+1=9$ **b.** $(8x+5)^{\frac{1}{3}}+2=-1$ **c.** $x^{\frac{2}{3}}-5=4$

## 2.3 Radicals

**$n^{th}$ Root of a number**

If $b^n = a$, then $b$ is an $n^{th}$ root of $a$. The principal $n^{th}$ root of $a$ is written $\sqrt[n]{a}$. $n$ is called the **index** of the radical, $a$ is called the **radicand** and $\sqrt{\ }$ is called the **radical**.

We use the term **square root** for $\sqrt{a}$ and **cube root** for $\sqrt[3]{a}$.

### 2.3.1 Converting expressions between radicals and exponents

***Rational Exponent*** $a^{\frac{1}{n}}$

If $\sqrt[n]{a}$ is a real number and $n \geq 2$, then $a^{\frac{1}{n}} = \sqrt[n]{a}$.

**Example 2.14: Write as a radical expression:**

**a.** $x^{\frac{1}{2}}$ **b.** $y^{\frac{1}{3}}$ **c.** $z^{\frac{1}{4}}$

**Solution:**

**a.** $x^{\frac{1}{2}} = \sqrt{x}$ **b.** $y^{\frac{1}{3}} = \sqrt[3]{y}$ **c.** $z^{\frac{1}{4}} = \sqrt[4]{z}$

In the next example, we will write each radical using a rational exponent.

**Example 2.15: Write with a rational exponent:**

**a.** $\sqrt{5y}$ **b.** $(\sqrt[3]{2x})^4$ **c.** $\sqrt{\left(\frac{3a}{4b}\right)^3}$

**Solution:**

**a.** $\sqrt{5y} = (5y)^{\frac{1}{2}}$ **b.** $(\sqrt[3]{2x})^4 = (2x)^{\frac{4}{3}}$ **c.** $\sqrt{\left(\frac{3a}{4b}\right)^3} = \left(\frac{3a}{4b}\right)^{\frac{3}{2}}$

First we define absolute value function before studying the properties of radicals.

***Absolute value function***

The absolute value function can be defined as a piecewise function $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

### 2.3.2 Properties of Radicals

**1. Definition of $\sqrt[n]{a}$**

***Properties of $\sqrt[n]{a}$***

When $n$ is an **even number** and

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When $n$ is an **odd number**, $\sqrt[n]{a}$ is a real number for all values of $a$.

We will apply these properties in the next two examples. Be alert for the negative signs as well as even and odd powers.

**Example 2.16: Simplify:**

**a.** $\sqrt[3]{64}$ **b.** $\sqrt[4]{81}$ **c.** $\sqrt[3]{-125}$ **c.** $\sqrt[4]{-16}$

**Solution:**

**a.** Since $4^3 = 64$, $\sqrt[3]{64} = 4$.

**b.** Since $3^4 = 81$, $\sqrt[4]{81} = 3$.

**c.** Since $(-5)^3 = -125$, $\sqrt[3]{-125} = -5$.

**d.** Think, $(?)^4 = -16$. No real number raised to the fourth power is negative. Hence $\sqrt[4]{-16}$ is not a real number.

## 2. Simplifying Odd and Even Roots

***Simplifying Odd and Even Roots***

For any integer $n \geq 2$,

- when the index $n$ is odd $\sqrt[n]{a^n} = a$.
- when the index $n$ is even $\sqrt[n]{a^n} = |a|$.

`We must use the absolute value signs when we take an even root of an expression with a variable in the radical.`

**Example 2.17: Simplify:**

**a.** $\sqrt{x^2}$ **b.** $\sqrt[3]{n^3}$ **c.** $\sqrt{x^6}$ **d.** $\sqrt[4]{z^8}$ **e.** $\sqrt[4]{16q^{12}}$

**Solution:**

**a.** We use the absolute value to be sure to get the positive root.Since the index $n = 2$ is even, $\sqrt{x^2} = |x|$.

**b.** This is an odd indexed root so there is no need for an absolute value sign. Hence $\sqrt[3]{n^3} = n$.

**c.** $\sqrt{x^6} = \sqrt{(x^3)^2} = |x^3|$. [Since the index is even]

**d.** $\sqrt[4]{z^8} = \sqrt[4]{(z^2)^4} = |z^2| = z^2$.

[`In this case the absolute value sign is not needed as` $z^2$ `is positive`].

**e.** $\sqrt[4]{16q^{12}} = \sqrt[4]{(2q^3)^4} = |2q^3| = 2|q^3|$.[Since the index is even]

## 3. Simplified Radical Expressions

A radical expression, $\sqrt[n]{a}$, is considered simplified if it has no factors of $m^n$.

***Simplified Radical Expression***

For real numbers $a$ and $m$, and an integer $n \geq 2$,

$\sqrt[n]{a}$ is considered simplified if $a$ has no factors of the form $m^n$.

For example, $\sqrt{6}$ is simplified because there are no perfect square factors in 6. But $\sqrt{24}$ is not simplified because 24 has a perfect square factor of 4. $24 = 6 \cdot 2^2$.

$\sqrt{24} = \sqrt{6 \cdot 2^2} = \sqrt{6}\sqrt{2^2} = |2|\sqrt{6} = 2\sqrt{6}$.

## 4. Use the Product Property to Simplify Radical Expressions

***Product Rule***

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and $n \geq 2$ is an integer, then

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} = \text{ and } \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

***How To* :: Simplify a Radical Expression using the Product Property**




Step 1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.

Step 2. Use the product rule to rewrite the radical as the product of two radicals.

Step 3. Simplify the root of the perfect power.

**Example 2.18: Simplify:**

**a.** $\sqrt{500}$ **b.** $\sqrt[4]{x^7}$ **c.** $\sqrt[3]{24x^7}$ **d.** $\sqrt{63u^3v^5}$

**Solution:**

**a.**

| | | |
|---|---|---|
| $\sqrt{500}$ | $=\sqrt{100 \cdot 5}$ | Rewrite the radicand as a product using the largest perfect square factor |
| | $=\sqrt{100} \cdot \sqrt{5}$ | Rewrite the radical as the product of two radicals |
| | $= 10\sqrt{5}$ | Simplify. |

**b.**

| | | |
|---|---|---|
| $\sqrt[4]{x^7}$ | $= \sqrt[4]{x^4 \cdot x^3}$ | Rewrite the radicand as a product using the greatest perfect fourth factor |
| | $= \sqrt[4]{x^4} \cdot \sqrt[4]{x^3}$ | Rewrite the radical as the product of two radicals |
| | $= \lvert x \rvert \sqrt[4]{x^3}$ | Simplify. |

**c.**

| | | |
|---|---|---|
| $\sqrt[3]{24x^7}$ | $= \sqrt[3]{8x^6 \cdot 3x}$ | Rewrite the radicand as a product using the greatest perfect cube factor |
| | $= \sqrt[3]{8x^6} \cdot \sqrt[3]{3x}$ | Rewrite the radical as the product of two radicals |
| | $= \sqrt[3]{(2x^2)^3} \cdot \sqrt[3]{3x}$ | Rewrite the first radicand as $(2x^2)^3$ |
| | $= 2x^2\sqrt[3]{3x}$ | Simplify. |

**d.**

| | | |
|---|---|---|
| $\sqrt{63u^3v^5}$ | $=\sqrt{9u^2v^4 \cdot 7uv}$ | Rewrite the radicand as a product using the largest perfect square factor |

$$
\begin{aligned}
&= \sqrt{9u^2v^4} \cdot \sqrt{7uv} && \text{Rewrite the radical as the product of two radicals} \\
&= \sqrt{(3uv^2)^2} \cdot \sqrt{7uv} && \text{Rewrite the first radicand as } (3uv^2)^2 \\
&= |3uv^2| \sqrt{7uv} && \text{Simplify.} \\
&= 3|u|v^2 \sqrt{7uv} && \text{Simplify.}
\end{aligned}
$$

## 5. Use the Quotient Property to Simplify Radical Expressions

***Quotient Rule***

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

***How To* :: Simplify a Radical Expression using the Quotient Property**

How?

Step 1. Simplify the fraction in the radicand, if possible.

Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.

Step 3. Simplify the radicals in the numerator and the denominator.

Whenever you have to simplify a radical expression, the first step you should take is to determine whether the radicand is a perfect power of the index. If not, check the numerator and denominator for any common factors, and remove them.

**Example 2.19: Simplify:**

**a.** $\sqrt[3]{\frac{16}{54}}$ **b.** $\sqrt[4]{\frac{a^{10}}{a^2}}$ **c.** $\sqrt{\frac{18p^5q^7}{32pq^2}}$ **d.** $\frac{\sqrt{48a^7}}{\sqrt{3a}}$

**Solution:**

**a.**

$$
\begin{aligned}
\sqrt[3]{\frac{16}{54}} &= \sqrt[3]{\frac{2 \cdot 8}{2 \cdot 27}} && \text{Rewrite showing the common factors of the numerator and denominator.} \\
&= \sqrt[3]{\frac{8}{27}} && \text{Simplify the fraction by removing common factors.} \\
&= \sqrt[3]{\frac{2^3}{3^3}} && \text{Rewrite the radicand as a quotient using the greatest perfect cube} \\
&= \frac{2}{3} && \text{Simplify.}
\end{aligned}
$$

**b.** $\sqrt[4]{\dfrac{a^{10}}{a^2}}$

$= \sqrt[4]{a^8}$ Use Quotient rule of exponent

$= \sqrt[4]{(a^2)^4}$ Rewrite the radicand as a quotient using the greatest perfect fourth power

$= |a^2|$ Since the index $n$ is even

$= a^2$

**c.** $\sqrt{\dfrac{18p^5q^7}{32pq^2}}$

$= \sqrt{\dfrac{9p^4q^5}{16}}$ Simplify the fraction in the radicand, if possible.

$= \dfrac{\sqrt{9p^4q^5}}{\sqrt{16}}$ Rewrite using the Quotient Property.

$= \dfrac{\sqrt{9p^4q^4 \cdot q}}{4}$ Rewrite each radicand as a product using perfect square factors.

$= \dfrac{\sqrt{9p^4q^4}\sqrt{q}}{4}$ Simplify the radicals in the numerator and the denominator.

$= \dfrac{3p^2q^2\sqrt{q}}{4}$ Simplify.

**d.** $\dfrac{\sqrt{48a^7}}{\sqrt{3a}}$

$= \sqrt{\dfrac{48a^7}{3a}}$ Use the Quotient Property to write as one radical.

$= \sqrt{16a^6}$ Simplify the fraction under the radical.

$= 4|a^3|$ Simplify.

### 2.3.3 Add, Subtract, and Multiply Radical Expressions

#### Add and Subtract Radical Expressions

**Like Radicals:** Like radicals are radical expressions with the **same index** and the **same radicand**.

We add and subtract like radicals in the same way we add and subtract like terms. When the radicals are not like, you cannot combine the terms.

**Example 2.20: Simplify:**

**a.** $2\sqrt{2} - 7\sqrt{2}$ **b.** $5\sqrt[3]{y} + 4\sqrt[3]{y}$ **c.** $\sqrt{20} + 3\sqrt{5}$ **d.** $7\sqrt[4]{x} - 2\sqrt[4]{y}$

**Solution**

**a.** $2\sqrt{2} - 7\sqrt{2} = -5\sqrt{2}$.

**b.** $5\sqrt[3]{y} + 4\sqrt[3]{y} = 9\sqrt[3]{y}$.

**c.** $\sqrt{20}+3\sqrt{5}=\sqrt{4\cdot 5}+3\sqrt{5}$ Simplify the radicals, when possible.

$=2\sqrt{5}+3\sqrt{5}$ Simplify.

$=5\sqrt{5}$ Combine the like radicals.

**d.** $7\sqrt[4]{x}-2\sqrt[4]{y}=7\sqrt[4]{x}-2\sqrt[4]{y}$. [*Since the radicals are not like, we cannot subtract them.*]

---

It is common practice at this point for us to ***assume all variables are greater than or equal to zero so that absolute values are not needed***. We will use this assumption throughout the rest of this chapter.

**Example 2.21: Simplify:**

**a.** $9\sqrt{50m^2}-6\sqrt{48m^2}$ **b.** $\sqrt{32x^7}-\sqrt{50x^7}$

**Solution:**

**a.** $9\sqrt{50m^2}-6\sqrt{48m^2}=9\sqrt{2\cdot 25m^2}-6\sqrt{3\cdot 16m^2}$ Simplify the radicals.

$=45m\sqrt{2}-24m\sqrt{3}$ The radicals are not like and so cannot be combined.

**b.** $\sqrt{32x^7}-\sqrt{50x^7}=\sqrt{2\cdot 16x^6\cdot x}-\sqrt{2\cdot 25x^6\cdot x}$ Simplify the radicals, when possible.

$=4x^3\sqrt{2x}-5x^3\sqrt{2x}$ Simplify.

$=-x^3\sqrt{2x}$ Combine the like radicals.

---

### Multiply Radical Expressions

Remember, we assume all variables are greater than or equal to zero. We will use the Distributive Property to multiply expressions with radicals and then simplify the radicals when possible.

**Example 2.22: Simplify:**

**a.** $(10\sqrt{6p^3})(4\sqrt{3p})$ **b.** $(3\sqrt{2}-\sqrt{5})(\sqrt{2}+4\sqrt{5})$

**Solution:**

**a.** $(10\sqrt{6p^3})(4\sqrt{3p})=40\sqrt{18p^4}$ Multiply using the Product Property.

$=40\sqrt{9p^4\cdot 2}$ Simplify the radical.

$=40\cdot 3p^2\sqrt{2}$ Simplify.

$=120p^2\sqrt{2}$

**b.** $(3\sqrt{2}-\sqrt{5})(\sqrt{2}+4\sqrt{5})=3\sqrt{2}\cdot\sqrt{2}+3\sqrt{2}\cdot 4\sqrt{5}-\sqrt{5}\cdot\sqrt{2}-\sqrt{5}\cdot 4\sqrt{5}$ Multiply.

$=3\cdot 2+12\sqrt{10}-\sqrt{10}-4\cdot 5$ Multiply.

| | |
|---|---|
| $= 6 + 12\sqrt{10} - \sqrt{10} - 20$ | Simplify. |
| $= -14 + 11\sqrt{10}$ | Combine like terms. |

We will use the special product formulas in the next few examples.
$(a+b)^2 = a^2 + 2ab + b^2, (a-b)^2 = a^2 - 2ab + b^2$ and $(a+b)(a-b) = a^2 - b^2$.

**Example 2.23: Simplify:**

**a.** $(2+\sqrt{3})^2$ **b.** $(4-2\sqrt{5})^2$ **c.** $(5-2\sqrt{3})(5+2\sqrt{3})$

**Solution:**

**a.** $(2+\sqrt{3})^2$

| | |
|---|---|
| $= 2^2 + 2 \cdot 2 \cdot \sqrt{3} + (\sqrt{3})^2$ | Multiply, using the Product of Binomial Squares Pattern |
| $= 4 + 4\sqrt{3} + 3$ | Simplify. |
| $= 7 + 4\sqrt{3}$ | Combine like terms. |

**b.** $(4-2\sqrt{5})^2$

| | |
|---|---|
| $= 4^2 - 2 \cdot 4 \cdot 2\sqrt{5} + (2\sqrt{5})^2$ | Multiply, using the Product of Binomial Squares Pattern |
| $= 16 - 16\sqrt{5} + 4 \cdot 5$ | Simplify. |
| $= 16 - 16\sqrt{5} + 20$ | Simplify. |
| $= 36 - 16\sqrt{5}$ | Combine like terms. |

**c.** $(5-2\sqrt{3})(5+2\sqrt{3})$

| | |
|---|---|
| $= 5^2 - (2\sqrt{3})^2$ | Multiply, using the Product of Conjugates Pattern. |
| $= 25 - 4 \cdot 3$ | Simplify. |
| $= 25 - 12$ | |
| $= 13$ | |

### 2.3.4 Rationalizing the Denominator

Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

**Simplified Radical Expressions**

***Simplified Radical Expressions***

A radical expression is considered simplified if there is

- no factors in the radicand have perfect powers of the index
- no fractions in the radicand
- no radicals in the denominator of a fraction

## 1. Rationalize a One Term Denominator

***How To* :: Rationalize a One Term Denominator**




- To rationalize a denominator with one term, we can multiply a square root by itself. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.
- To rationalize a denominator with a higher index radical, we multiply the numerator and denominator by a radical that would give us a radicand that is a perfect power of the index.

To rationalize a denominator with a square root, we use the property that $(\sqrt{a})^2 = a$.

**Example 2.24: Rationalize the denominator: a.** $\dfrac{4}{\sqrt{3}}$ **b.** $\dfrac{3}{\sqrt{6x}}$

**Solution:**

**a.**
$$\frac{4}{\sqrt{3}} = \frac{4 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$$ Multiply both the numerator and denominator by $\sqrt{3}$.
$$= \frac{4\sqrt{3}}{3}$$ Simplify.

**b.**
$$\frac{3}{\sqrt{6x}} = \frac{3 \cdot \sqrt{6x}}{\sqrt{6x} \cdot \sqrt{6x}}$$ Multiply the numerator and denominator by $\sqrt{6x}$.
$$= \frac{3\sqrt{6x}}{6x}$$ Simplify.
$$= \frac{\sqrt{6x}}{2x}$$ Simplify.

## 2. Rationalize a Two Term Denominator

When the denominator of a fraction is a sum or difference with square roots, we use the Product of Conjugates Pattern to rationalize the denominator.

**Example 2.25: Rationalize the denominator of** $\dfrac{5}{2-\sqrt{3}}$

**Solution:** Multiply the numerator and denominator by the conjugate of the denominator.

$$\frac{5}{2-\sqrt{3}} = \frac{5(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$$
$$= \frac{5(2+\sqrt{3})}{2^2-(\sqrt{3})^2}$$ Multiply the conjugates in the denominator.
$$= \frac{5(2+\sqrt{3})}{4-3}$$ Simplify the denominator.

$$= \frac{5(2+\sqrt{3})}{1} \qquad \text{Simplify the denominator.}$$

$$= 5(2+\sqrt{3}) \qquad \text{Simplify.}$$

**Example 2.26: Rationalize the denominator of** $\dfrac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$

**Solution:** Multiply the numerator and denominator by the conjugate of the denominator.

$$\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}} = \frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x}-\sqrt{7})(\sqrt{x}+\sqrt{7})}$$

$$= \frac{(\sqrt{x}+\sqrt{7})^2}{(\sqrt{x})^2-(\sqrt{7})^2} \qquad \text{Multiply the conjugates in the denominator.}$$

$$= \frac{(\sqrt{x}+\sqrt{7})^2}{x-7} \qquad \text{Simplify the denominator.}$$

We do not square the numerator. Leaving it in factored form, we can see there are no common factors to remove from the numerator and denominator.

### 2.3.5 Solve Radical Equations

In this section we will solve equations that have a variable in the radicand of a radical expression. An equation of this type is called a radical equation.

**Radical Equation:** An equation in which a variable is in the radicand of a radical expression is called a radical equation.

**Example 2.27:** Solve: $\sqrt{5n-4}-9=0$

**Solution:**

| | | |
|---|---|---|
| Isolate the radical on one side of the equation. | To isolate the radical add 9 to both sides.<br>Simplify | $\sqrt{5n-4}-9=0$<br>$\sqrt{5n-4}-9+9=0+9$<br>$\sqrt{5n-4}=9$ |
| Raise both sides of the equation to the power of the index. | Since the index is 2, square both sides. | $(\sqrt{5n-4})^2=9^2$ |
| Solve the new equation. | Remember $(\sqrt{a})^2=a$. | $5n-4=81$<br>$5n=85$<br>$n=17$ |

***How To* :: Solve a Radical Equation with One Radical.**




Step 1. Isolate the radical on one side of the equation.

Step 2. Raise both sides of the equation to the power of the index.

Step 3. Solve the new equation.

Step 4. Check the answer in the original equation.

**When the index of the radical is 3, we cube both sides to remove the radical.**

**Example 2.28: Solve:** $\sqrt[3]{5x+1}+8=4$

**Solution:**

| | |
|---|---|
| $\sqrt[3]{5x+1}=-4$ | To isolate the radical, subtract 8 from both sides. |
| $(\sqrt[3]{5x+1})^3=(-4)^3$ | Cube both sides. |
| $5x+1=-64$ | Simplify. |
| $5x=-65$ | Solve the equation. |
| $x=-13$ | |

## 2.3 Section Exercises

**Simplify Expressions with Roots**

*In the following exercises, simplify.*

**1. a.** $\sqrt{64}$ **b.** $-\sqrt{81}$ **c.** $\sqrt[5]{-32}$ **d.** $\sqrt{\frac{64}{121}}$

**Simplify Variable Expressions with Roots**

*In the following exercises, simplify using* ***absolute values*** *as necessary.*

**2. a.** $\sqrt{x^6}$ **b.** $\sqrt[3]{-8c^9}$ **c.** $\sqrt[4]{16x^8}$ **d.** $\sqrt[5]{a^{10}}$

*In the following exercises, write as a radical expression.*

**3. a.** $x^{\frac{1}{2}}$ **b.** $c^{\frac{1}{9}}$ **c.** $(2x^5y^2)^{\frac{1}{3}}$ **d.** $(2xy^2)^{\frac{1}{5}}$

*In the following exercises, write with a rational exponent.*

**4. a.** $\sqrt[7]{x}$ **b.** $\sqrt[4]{5x}$ **c.** $\sqrt[4]{\left(\frac{2xy}{5z}\right)^2}$ **d.** $\sqrt{x^5}$

**Use the Product Property to Simplify Radical Expressions**

*In the following exercises, simplify using absolute value signs as needed.*

**5. a.** $\sqrt{27}$ **b.** $\sqrt[4]{48y^{16}}$ **c.** $\sqrt[4]{32x^5y^4}$ **d.** $\sqrt[3]{81p^7q^8}$

### Use the Quotient Property to Simplify Radical Expressions

*In the following exercises, use the Quotient Property to simplify.*

**6.** **a.** $\sqrt{\frac{45}{80}}$ **b.** $\sqrt{\frac{28p^7}{q^2}}$ **c.** $\sqrt[3]{\frac{24x^8y^4}{81x^2y}}$ **d.** $\sqrt[4]{\frac{6}{96}}$

### Add and Subtract Radical Expressions

*In the following exercises, simplify*

**7.** **a.** $\sqrt{48}-\sqrt{27}$ **b.** $\sqrt[3]{24}-\sqrt[3]{81}$ **c.** $8\sqrt[3]{64q^6}-3\sqrt[3]{125q^6}$

### Multiply Radical Expressions

*In the following exercises, simplify*

**8.** **a.** $(-2\sqrt{3})(3\sqrt{18})$ **b.** $\sqrt{7}(5+2\sqrt{7})$ **c.** $(3-2\sqrt{7})(5-4\sqrt{7})$

**9.** **a.** $(3+\sqrt{5})^2$ **b.** $(2-5\sqrt{3})^2$ **c.** $(4+\sqrt{2})(4-\sqrt{2})$

### Rationalize a One Term Denominator

*In the following exercises, rationalize the denominator.*

**10.** **a.** $\frac{10}{\sqrt{6}}$ **b.** $\sqrt{\frac{4}{27}}$ **c.** $\frac{6}{\sqrt[4]{9x^3}}$

### Rationalize a Two Term Denominator

*In the following exercises, rationalize the denominator.*

**11.** **a.** $\frac{8}{1-\sqrt{5}}$ **b.** $\frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$ **c.** $\frac{3\sqrt{2}+2\sqrt{3}}{5\sqrt{2}-3\sqrt{3}}$

### Solve Radical Equations

*In the following exercises, solve.*

**12.** **a.** $2\sqrt{5x+1}-8=0$ **b.** $\sqrt[3]{4x+5}-2=-5$ **c.** $\sqrt{6n+1}+4=8$

# Chapter 3

# POLYNOMIALS

## Contents


**3.1 Addition and Subtraction of Polynomials**

3.1.1 Determine the Degree of Polynomials

3.1.2 Add and Subtract Polynomials

**3.2 Multiplication of Polynomials**

3.2.1 Multiplication of Polynomials

3.2.2 Special Products


**Learning outcome covered:**

**n.** `Perform operations on polynomials and manipulate numerical and polynomial expressions and solve first degree equations.`

**Learning Objectives**

**At the end of this chapter, the students will be able to:**

- Determine the degree of polynomials
- Add and subtract polynomials
- Multiply a polynomial by another polynomial
- Simplify special products

## Introduction

A polynomial is a special algebraic expression and is used widely in algebra. In this chapter you will investigate polynomials and polynomial functions and learn how to perform mathematical operations on them.

## 3.1 Addition and Subtraction of Polynomials

Let us start with some basic concepts.

**Term**

A **term** is a constant or the product of a constant and one or more variables.

Examples of terms are 7, $y$, $5x^2$, $9a$, and $b^5$.

**Coefficient**

The **coefficient** of a term is the constant that multiplies the variable in a term.

Think of the coefficient as the number in front of the variable. The coefficient of the term $3x$ is 3. When we write $x$, the coefficient is 1, since $x = 1 \cdot x$.

**Like Terms**

Terms that are either constants or have the same variables raised to the same powers are called **like terms**.

$5x$ and $3x$ are like terms where as $2x^2y$ and $3xy^2$ are not like terms.

If there are like terms in an expression, you can simplify the expression by combining the like terms. We add the coefficients and keep the same variable. The following example shows how to combine like terms.

$$\begin{aligned} 2x^2 + 3x + 7 + x^2 + 4x + 5 &= 2x^2 + x^2 + 3x + 4x + 7 + 5 \\ &= 3x^2 + 7x + 12 \end{aligned}$$

### 3.1.1 Determine the Degree of Polynomials

We have learned that a term is a constant or the product of a constant and one or more variables. A monomial is an algebraic expression with one term. When it is of the form $ax^m$, where $a$ is a constant and $m$ is a whole number, it is called a monomial in one variable. Some examples of monomial in one variable are $x$, $4y^2$, $3a$. Monomials can also have more than one variable such as $-4a^2b^3c^2$.

**Monomial**

A monomial is an algebraic expression with one term.
A monomial in one variable is a term of the form $ax^m$, where $a$ is a constant and $m$ is a whole number.

**Polynomials**

**polynomial** - A monomial, or two or more monomials, combined by addition or subtraction

**monomial** - A polynomial with exactly one term

**binomial** - A polynomial with exactly two terms

**trinomial** - A polynomial with exactly three terms

Here are some examples of polynomials:

| | | | | |
|---|---|---|---|---|
| Polynomial | $y+1$ | $4a^2-7ab+2b^2$ | $4x^4+x^3+8x^2-9x+1$ | |
| Monomial | 14 | $8y^2$ | $-9x^3y^5$ | $-13a^3b^2c$ |
| Binomial | $a+7b$ | $4x^2-y^2$ | $y^2-16$ | $3p^3q-9p^2q$ |
| Trinomial | $x^2-7x+12$ | $9m^2+2mn-8n^2$ | $6k^4-k^3+8k$ | $z^4+3z^2-1$ |

We use the words monomial, binomial, and trinomial when referring to these special polynomials and just call all the rest polynomials.

The **degree of a polynomial** and the degree of its terms are determined by the exponents of the variable.

A monomial that has no variable, just a constant, is a special case. The **degree of a constant term** is 0.

**Degree of a Polynomial**

The **degree of a term** is the sum of the exponents of its variables.

The **degree of a constant** is 0.

The **degree of a polynomial** is the highest degree of all its terms.

Let's start by looking at a monomial. The monomial $8ab^2$ has two variables $a$ and $b$. To find the degree we need to find the sum of the exponents. The variable $a$ doesn't have an exponent written, but remember, that means the exponent is 1. The exponent of $b$ is 2. The sum of the exponents, 1 + 2, is 3 so the degree is 3.

Here are some additional examples.

| | | | | |
|---|---|---|---|---|
| **Monomials** | $14$ | $8ab^2$ | $-9x^3y^5$ | $-13a$ |
| Degree | 0 | 3 | 8 | 1 |
| **Binomials** | $h+7$ | $7b^2-3b$ | $x^2y^2-25$ | $4n^3-8n^2$ |
| Degree of each term | 1 0 | 2 1 | 4 0 | 3 2 |
| Degree of polynomial | 1 | 2 | 4 | 3 |
| **Trinomial** | $x^2-12x+7$ | $9a^2-6ab+b^2$ | $6m^4-m^3n^2+8mn^5$ | $z^4+3z^2-1$ |
| Degree of each term | 2 1 0 | 2 2 2 | 4 5 6 | 4 2 0 |
| Degree of polynomial | 2 | 2 | 6 | 4 |
| **Polynomial** | $y-1$ | $3y^2-2y-5$ | $4x^4+x^3+8x^2-9x+1$ | |
| Degree of each term | 1 0 | 2 1 0 | 4 3 2 1 0 | |
| Degree of polynomial | 1 | 2 | 4 | |

**Example 3.1: Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. Then, find the degree of each polynomial.**

**a.** $7y^2-5y+3$ **b.** $-2a^4b^2$ **c.** $3x^5-4x^3-6x^2+x-8$ **d.** $2y-8xy^3$ **e.** $15$

**Solution:**

| | Polynomial | Number of terms | Type | Degree of terms | Degree of polynomial |
|---|---|---|---|---|---|
| **a.** | $7y^2-5y+3$ | 3 | Trinomial | 2, 1, 0 | 2 |
| **b.** | $-2a^4b^2$ | 1 | Monomial | 6 | 6 |
| **c.** | $3x^5-4x^3-6x^2+x-8$ | 5 | Polynomial | 5, 3, 2, 1, 0 | 5 |
| **d.** | $2y-8xy^3$ | 2 | Binomial | 1, 4 | 4 |
| **e.** | 15 | 1 | Monomial | 0 | 0 |

Working with polynomials is easier when you list the terms in descending order of degrees. When a polynomial is written this way, it is said to be in **standard form of a polynomial**. Get in the habit of writing the term with the highest degree first.

### 3.1.2 Add and Subtract Polynomials

**Example 3.2:** **Simplify:**

**a.** $a^2 + 7b^2 - 6a^2$ **b.** $u^2v + 5u^2 - 3v^2$

**Solution:**

**a.** $a^2 + 7b^2 - 6a^2 = -5a^2 + 7b^2$ Combine like terms.

**b.** $u^2v + 5u^2 - 3v^2 = u^2v + 5u^2 - 3v^2$ There are no like terms to combine.

---

We can think of adding and subtracting polynomials as just adding and subtracting a series of monomials. Look for the like termsthose with the same variables and the same exponent. The Commutative Property allows us to rearrange the terms to put like terms together.

**Example 3.3:** Find the sum: $(7y^2 - 2y + 9) + (4y^2 - 8y - 7)$

**Solution:**

$$\begin{aligned}(7y^2 - 2y + 9) + (4y^2 - 8y - 7) &= (7y^2 - 2y + 9) + (4y^2 - 8y - 7) && \text{Identify like terms.}\\ &= 7y^2 + 4y^2 - 2y - 8y + 9 - 7 && \text{Rewrite without parentheses}\\ &= 11y^2 - 10y + 2 && \text{Combine like terms.}\end{aligned}$$

---

***Be careful with the signs as you distribute while subtracting the polynomials in the next example.***

**Example 3.4:** Find the difference: $(9w^2 - 7w + 5) - (2w^2 - 4)$

**Solution:**

$$\begin{aligned}(9w^2 - 7w + 5) - (2w^2 - 4) &= 9w^2 - 7w + 5 - 2w^2 + 4 && \text{Distribute and identify like terms.}\\ &= 9w^2 - 2w^2 - 7w + 5 + 4 && \text{Rearrange the terms.}\\ &= 7w^2 - 7w + 9 && \text{Combine like terms.}\end{aligned}$$

---

***To subtract*** $a$ ***from*** $b$***, we write it as*** $b - a$***, placing the*** $b$ ***first.***

**Example 3.5:** Subtract $p^2 + 10pq - 2q^2$ from $p^2 + q^2$.

**Solution:**

$$\begin{aligned}(p^2 + q^2) - (p^2 + 10pq - 2q^2) &= p^2 + q^2 - p^2 - 10pq + 2q^2 && \text{Distribute.}\\ &= p^2 - p^2 - 10pq + q^2 + 2q^2 && \text{Rearrange the terms, to put like terms together.}\\ &= -10pq + 3q^2 && \text{Combine like terms.}\end{aligned}$$

## 3.1 Section Exercises

### Determine the Type of Polynomials

*In the following exercises, determine if the polynomial is a monomial, binomial, trinomial, or other polynomial.*

**1.**

a. $47x^5-17x^2y^3+y^2$

b. $5c^3+11c^2-c-8$

**2.**

a. $x^2-y^2$

b. $-13c^4$

**3.**

a. $8y-5x$

b. $y^2-5yz-6z^2$

### Add and Subtract Polynomials

*In the following exercises, add the polynomials.*

**4.** $(5y^2+12y+4)+(6y^2-8y+7)$

**5.** $(x^2+6x+8)+(-4x^2+11x-9)$

**6.** $(8x^2-5x+2)+(3x^2+3)$

**7.** $(5a^2+8)+(a^2-4a-9)$

*In the following exercises, subtract the polynomials.*

**8.** $(4m^2-6m-3)-(2m^2+m-7)$

**9.** $(3b^2-4b+1)-(5b^2-b-2)$

**10.** $(12s^2-15s)-(s-9)$

**11.** $(a^2+8a+5)-(a^2-3a+2)$

*In the following exercises, subtract the polynomials.*

**12.** Subtract $(9x^2+2)$ from $(12x^2-x+6)$

**13.** Subtract $(5y^2-y+12)$ from $(5y^2-8y-20)$

**14.** Subtract $(7w^2-4w+2)$ from $(8w^2-w+6)$

**15.** Subtract $(5x^2-x+12)$ from $(9x^2-6x-20)$

*In the following exercises, add or subtract the polynomials.*

**16.** $(p^3-3p^2q)+(2pq^2+4q^3)-(3p^2q+pq^2)$

**17.** $(a^3-2a^2b)+(ab^2+b^3)-(3a^2b+4ab^2)$

**18.** $(x^3-x^2y)-(4xy^2-y^3)+(3x^2y-xy^2)$

**19.** $(x^3-2x^2y)-(xy^2-3y^3)-(x^2y-4xy^2)$

**20.** Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. Then, find the degree of each polynomial.

| | Polynomial | Number of terms | Type | Degree of polynomial |
|---|---|---|---|---|
| **a.** | $3x^2 - 5xy + 3xy^2$ | | | |
| **b.** | $-5xy^2$ | | | |
| **c.** | $5x^3y^2z^3 + y^5z^2$ | | | |
| **d.** | $5x - 1$ | | | |
| **e.** | $x^3 - 3x^2y + 3xy^3 - y^3$ | | | |
| **f.** | $2$ | | | |

## 3.2 Multiplication of Polynomials

### 3.2.1 Multiplication of Polynomials

#### Multiply Monomials

We are ready to perform operations on polynomials. Since monomials are algebraic expressions, we can use the properties of exponents to multiply monomials.

**Example 3.6:** **Multiply:** **a.** $(3x^2)(-4x^3)$ **b.** $\left(\frac{5}{6}x^3y\right)(2xy^2)$

**Solution:**

**a.** $(3x^2)(-4x^3)$

$= 3 \cdot (-4) \cdot x^2 \cdot x^3$ Rearrange the terms.

$= -12x^5$ Multiply.

**b.** $\left(\frac{5}{6}x^3y\right)(12xy^2)$

$= \frac{5}{6} \cdot 12 \cdot x^3 \cdot x \cdot y \cdot y^2$ Rearrange the terms.

$= 10x^4y^3$ Multiply.

## Multiply a Polynomial by a Monomial

Multiplying a polynomial by a monomial is really just applying the Distributive Property.

**Example 3.7: Multiply: a.** $-2y(4y^2+3y-5)$ **b.** $3x^3y(x^2-8xy+y^2)$

**Solution:**

**a.** $-2y(4y^2+3y-5)$

$= -2y\cdot(4y^2)+(-2y)\cdot(3y)-(-2y)\cdot(5)$ Distribute.

$= -8y^3-6y^2+10y$ Multiply.

**b.** $3x^3y(x^2-8xy+y^2)$

$= 3x^3y\cdot x^2-(3x^3y)\cdot(8xy)+(3x^3y)\cdot y^2$ Distribute.

$= 3x^5y-24x^4y^2+3x^3y^3$ Multiply.

## Multiply a Binomial by a Binomial

Just like there are different ways to represent multiplication of numbers, there are several methods that can be used to multiply a binomial times a binomial. We will start by using the Distributive Property.

**Example 3.8: Multiply: a.** $(y+5)(y+8)$ **b.** $(4y+3)(2y-5)$

**Solution:**

**a.** $(y+5)(y+8)$

$(y+5)(y+8) = y(y+8)+5(y+8)$ Distribute (y+8).

$= y^2+8y+5y+40$ Distribute again.

$= y^2+13y+40$ Combine like terms.

**b.** $(4y+3)(2y-5)$

$(4y+3)(2y-5) = 4y(2y-5)+3(2y-5)$ Distribute (y+8).

$= 8y^2-20y+6y-15$ Distribute again.

$= 8y^2-14y-15$ Combine like terms.

## F O I L

If you multiply binomials often enough you may notice a pattern.

- Notice that the first term in the result is the product of the first term in each binomial.
- The second and third terms are the product of multiplying the two outer terms and then the two inner terms.
- And the last term results from multiplying the two last terms.
  *We abbreviate First, Outer, Inner, Last as* ***FOIL****.*

**Example 3.9:** Multiply: $(y-7)(y+4)$

**Solution:**

| | | |
|---|---|---|
| Step 1. Multiply the First terms. | $(y-7)(y+4)$ | $=y^2$ |
| Step 2. Multiply the Outer terms. | $(y-7)(y+4)$ | $=y^2+4y$ |
| Step 3. Multiply the Inner terms. | $(y-7)(y+4)$ | $=y^2+4y-7y$ |
| Step 4. Multiply the Last terms. | $(y-7)(y+4)$ | $=y^2+4y-7y-28$ |
| Step 5. Combine like terms. | | $=y^2-3y-28$ |

**Multiply a Polynomial by a Polynomial**

Now were ready to multiply a polynomial by a polynomial.

**Example 3.10:** Multiply $(b+3)(2b^2-5b+8)$.

**Solution:**

$$(b+3)(2b^2-5b+8) = b\cdot(2b^2)-b\cdot(5b)+b\cdot 8+3\cdot(2b^2)+3\cdot(-5b)+3\cdot 8 \quad \text{Distribute.}$$

$$= 2b^3-5b^2+8b+6b^2-15b+24 \quad \text{Multiply.}$$

$$= 2b^3+b^2-7b+24 \quad \text{Combine like terms.}$$

### 3.2.2 Special Products

Mathematicians like to look for patterns that will make their work easier. A good example of this is squaring binomials. While you can always get the product by writing the binomial twice and multiplying them, there is less work to do if you learn to use a pattern.

**Binomial Squares Pattern**

If $a$ and $b$ are real numbers,

- $(a+b)^2 = a^2+2ab+b^2$
- $(a-b)^2 = a^2-2ab+b^2$

***To square a binomial, square the first term, square the last term , double their product.***

**Example 3.11:** **Multiply:** **a.** $(x+5)^2$ **b.** $(2x-3y)^2$

**Solution:**

**a.** $(x+5)^2$

$= x^2 + 2 \cdot x \cdot 5 + 5^2$ $\quad (a+b)^2 = a^2 + 2ab + b^2$

$= x^2 + 10x + 25$ Simplify.

**b.** $(2x-3y)^2$

$= (2x)^2 - 2 \cdot 2x \cdot 3y + (3y)^2$ $\quad (a-b)^2 = a^2 - 2ab + b^2$

$= 4x^2 - 12xy + 9y^2$ Simplify.

---

A pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference is called a **conjugate pair** and is of the form $(a-b)$, $(a+b)$.

The product of conjugates is always of the form $a^2-b^2$. This is called a **difference of squares**. This leads to the pattern:

**Product of Conjugates Pattern**

If $a$ and $b$ are real numbers,

$$(a-b)(a+b) = a^2 - b^2.$$

**Example 3.12: Multiply using the product of conjugates pattern:**

**a.** $(2x+5)(2x-5)$ **b.** $(5m-9n)(5m+9n)$

**Solution:**

**a.** $(2x+5)(2x-5)$

$= (2x)^2 - 5^2$ $\quad (a-b)(a+b) = a^2 - b^2$

$= 4x^2 - 25$ Simplify.

**b.** $(5m-9n)(5m+9n)$

$= (5m)^2 - (9n)^2$ $\quad (a-b)(a+b) = a^2 - b^2$

$= 25m^2 - 81n^2$ Simplify.

---

## 3.2 Section Exercises

**Multiply Monomials** *In the following exercises, multiply the monomials.*

**1.** $(6y^7)(-3y^4)$

**2.** $(-10x^5)(-3x^3)$

**3.** $(-8u^6)(-9u)$

**4.** $\left(\frac{4}{7}rs^2\right)(14rs^3)$

**5.** $\left(\frac{5}{8}x^3y\right)(24x^5y)$

**6.** $\left(\frac{2}{3}x^2y\right)\left(\frac{3}{4}xy^2\right)$

**Multiply a Polynomial by a Monomial**

*In the following exercises, multiply.*

**7.** $-8x(x^2+2x-15)$

**8.** $-5t(t^2+3t-18)$

**9.** $-8y(y^2+2y-15)$

**10.** $5pq^3(p^2-2pq+6q^2)$

**11.** $9r^3s(r^2-3rs+5s^2)$

**12.** $-4y^2z^2(3y^2+12yz-z^2)$

**Multiply a Binomial by a Binomial**

*In the following exercises, multiply the binomials.*

**13.** $(w+5)(w+7)$

**14.** $(2xy+3)(3xy+2)$

**15.** $(4p+11)(5p-4)$

**16.** $(7q+4)(3q-8)$

**17.** $(x^2+3)(x+2)$

**18.** $(y^2-4)(y+3)$

**Multiply a Polynomial by a Polynomial**

*In the following exercises, multiply the polynomials.*

**19.** $(x+5)(x^2+4x+3)$

**20.** $(u+4)(u^2+3u+2)$

**21.** $(y+8)(4y^2+y-7)$

**22.** $(a+10)(3a^2+a-5)$

**23.** $(y^2-3y+8)(4y^2+y-7)$

### Multiply Special Products

*In the following exercises, square each binomial using the Binomial Squares Pattern.*

**24.** $(w+4)^2$

**25.** $(q+12)^2$

**26.** $(3x-5)^2$

**27.** $(2y-3z)^2$

**28.** $\left(y+\frac{1}{4}\right)^2$

**29.** $(3x^2+2)^2$

*In the following exercises, multiply each pair of conjugates using the Product of Conjugates Pattern.*

**30.** $(5k+6)(5k-6)$

**31.** $(9c+5)(9c-5)$

**32.** $(7w+10x)(7w-10x)$

# Chapter 4

# FACTORING POLYNOMIALS

## Contents


**4.1 Greatest Common Factor and Factor by Grouping**

4.1.1 Factor the Greatest Common Factor from a Polynomial

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**4.2 Factor Trinomials**

4.2.1 Factor Trinomials of the Form $x^2 + bx + c$

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**4.3 Factor Special Products**

4.3.1 Factor Perfect Square Trinomials

4.3.2 Factor Differences of Squares

4.3.3 Factor sums and differences of cubes


**Learning outcome covered:**

**n.** `Perform operations on polynomials and manipulate numerical and polynomial expressions and solve first degree equations.`

**Learning Objectives**

**At the end of this chapter, the students will be able to:**

- Find the greatest common factor of two or more expressions
- Factor the greatest common factor from a polynomial and by grouping
- Factor trinomials of the form $x^2 + bx + c$ and $ax^2 + bx + c$
- Factor perfect square trinomials
- Factor differences of squares
- Factor sums and differences of cubes

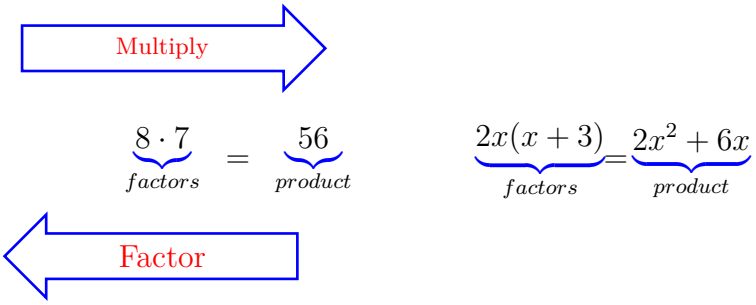
# Introduction

An epidemic of a disease has broken out. Where did it start? How is it spreading? What can be done to control it? Answers to these and other questions can be found by scientists known as epidemiologists. They collect data and analyze it to study disease and consider possible control measures. Because diseases can spread at alarming rates, these scientists must use their knowledge of mathematics involving **factoring**. In this chapter, you will learn how to factor and apply factoring to real-life situations.

## 4.1 Greatest Common Factor and Factor by Grouping

### Find the Greatest Common Factor of Two or More Expressions

Earlier we multiplied factors together to get a product. Now, we will reverse this process; we will start with a product and then break it down into its factors. Splitting a product into factors is called **factoring**.




**Greatest Common Factor**

The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

We summarize the steps we use to find the greatest common factor.

***How To* :: Find the greatest common factor (GCF) of two expressions**

How?

Step 1. Factor each coefficient into primes. Write all variables with exponents in expanded form.

Step 2. List all factorsmatching common factors in a column. In each column, circle the common factors.

Step 3. Bring down the common factors that all expressions share.

Step 4. Multiply the factors.

The next example will show us the steps to find the greatest common factor of three expressions.

**Example 4.1: Find the greatest common factor of $21x^3$, $9x^2$, $15x$.**

**Solution:**

| | |
|---|---|
| Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column. | $21x^3 = 3 \cdot 7 \cdot x \cdot x \cdot x$<br>$9x^2 = 3 \cdot 3 \cdot x \cdot x$<br>$15x = 3 \cdot 5 \cdot x$ |
| Bring down the common factors. | $\text{GCF} = 3 \cdot x$ |
| Multiply the factors. | $\text{GCF} = 3x$ |

The GCF of $21x^3$ , $9x^2$ and $15x$ is $3x$.

### 4.1.1 Factor the Greatest Common Factor from a Polynomial

It is sometimes useful to represent a number as a product of factors, for example, 12 as $2 \times 6$ or $3 \times 4$. In algebra, it can also be useful to represent a polynomial in factored form. We will start with a product, such as $3x^2 + 15x$, and end with its factors, $3x(x + 5)$. To do this we apply the Distributive Property in reverse.

***How To* :: Factor the greatest common factor from a polynomial.**

How?

Step 1. Find the GCF of all the terms of the polynomial.
Step 2. Rewrite each term as a product using the GCF.
Step 3. Use the reverse Distributive Property to factor the expression.
Step 4. Check by multiplying the factors.

**Example 4.2: Factor:** $8m^3 - 12m^2n + 20mn^2$

**Solution:**

| | | |
|---|---|---|
| Step 1: Find the GCF of all the terms of the polynomial. | Find the GCF of $8m^3$, $12m^2n$, $20mn^2$ | $8m^3 = 2 \cdot 2 \cdot 2 \cdot m \cdot m \cdot m$<br>$12m^2n = 2 \cdot 2 \cdot 3 \cdot m \cdot m \cdot n$<br>$20mn^2 = 2 \cdot 2 \cdot 5 \cdot m \cdot n \cdot n$<br>$GCF = 2 \cdot 2 \cdot m$<br>$GCF = 4m$ |
| Step 2: Rewrite each term as a product using the GCF. | Rewrite $8m^3$, $12m^2n$, $20mn^2$ as product of their GCF, $4m$. $8m^3 = 4m \cdot 2m^2$<br>$12m^2n = 4m \cdot 3mn$<br>$20mn^2 = 4m \cdot 5n^2$ | $8m^3 - 12m^2n + 20mn^2 =$<br>$4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$ |

| | | |
|---|---|---|
| Step 3: Use the reverse Distributive Property to factor the expression. | | $4m(2m^2 - 3mn + 5n^2)$ |
| Step 4: Check. | | $4m(2m^2 - 3mn + 5n^2) = 8m^3 - 12m^2n + 20mn^2$ |

**Example 4.3:** **Factor:** $8x^3y - 10x^2y^2 + 12xy^3$

**Solution:**

The GCF of $8x^3y$, $-10x^2y^2$, and $12xy^3$ is $2xy$.

$$\begin{aligned} 8x^3y &= 2\cdot 2\cdot 2 \quad x\cdot x\cdot x\cdot y \\ 10x^2y^2 &= 2\cdot \quad 5\cdot x\cdot x\cdot \quad y\cdot y \\ 12xy^3 &= 2\cdot 2\cdot 3 \quad x\cdot \quad y\cdot y\cdot y \\ \hline \text{GCF} &= 2\cdot \quad x\cdot \quad y \\ \text{GCF} &= 2xy \end{aligned}$$

$$\begin{aligned} 8x^3y - 10x^2y^2 + 12xy^3 &= 2xy\cdot 4x^2 - 2xy\cdot 5xy + 2xy\cdot 6y^2 \\ &= 2xy(4x^2 - 5xy + 6y^2) \end{aligned}$$

*So far our greatest common factors have been monomials. In the next example, the greatest common factor is a binomial.*

**Example 4.4:** **Factor:** $3y(y+7) - 4(y+7)$

**Solution:**

$3y(y+7) - 4(y+7) = 3y(y+7) - 4(y+7)$ The GCF is the binomial $y+7$.

$= (y+7)(3y-4)$ Factor the GCF, $(y+7)$.

### 4.1.2 Factor by Grouping

Sometimes there is no common factor of all the terms of a polynomial. When there are four terms we separate the polynomial into two parts with two terms in each part. Then look for the GCF in each part.

***How To* :: Factor by grouping.**




Step 1. Group terms with common factors.

Step 2. Factor out the common factor in each group.

Step 3. Factor the common factor from the expression.

Step 4. Check by multiplying the factors.

**Example 4.5: Factor by grouping:** $xy + 3y + 2x + 6$

**Solution:**

| | | |
|---|---|---|
| Step 1: Group terms with common factors. | Is there a gcf for all four terms?<br>No, so let us separate the first two terms from the second two. | $xy + 3y + 2x + 6 =$<br>$\underbrace{xy + 3y} + \underbrace{2x + 6}$ |
| Step 2: Factor out the common factor in each group. | Factor the GCF from the first two terms.<br>Factor the GCF from the second two terms. | $y(x + 3) + 2x + 6$<br>$y(x + 3) + 2(x + 3)$ |
| Step 3: Factor the common factor from the expression. | Notice that each term has a common factor of $(x + 3)$ | $y(x + 3) + 2(x + 3) =$<br>$(x + 3)(y + 2)$ |
| Step 4: Check by multiplying the factors. | Multiply $(x + 3)(y + 2)$. Is the product the original expression? | $(x+3)(y+2) = xy+2x+3y+6$<br>$= xy + 3y + 2x + 6$ |

**Example 4.6: Factor by grouping:**

**a.** $x^2 - 3x + 2x - 6$ **b.** $6x^2 - 3x - 4x + 2$.

**Solution:** There is no GCF in all four terms.

**a.**

| | | |
|---|---|---|
| $x^2 - 3x + 2x - 6$ | $= x^2 - 3x \quad + 2x - 6$ | Separate into two parts. |
| | $= x(x + 3) + 2(x - 3)$ | Factor the GCF from both parts. |
| | $= (x - 3)(x - 2)$ | Factor out the common factor. |

**b.**

| | | |
|---|---|---|
| $6x^2 - 3x - 4x + 2$ | $= 6x^2 - 3x \quad - 4x + 2$ | Separate into two parts. |
| | $= 3x(2x - 1) - 2(2x - 1)$ | Factor the GCF from both parts. Be careful with the signs when factoring the GCF from the last two terms. |
| | $= (2x - 1)(3x - 2)$ | Factor out the common factor. |

## 4.1 Section Exercises

### Find the Greatest Common Factor of Two or More Expressions

*In the following exercises, find the greatest common factor.*

**1.** $10p^3q, 12pq^2$

**2.** $8a^2b^3, 10ab^2$

**3.** $12m^2n^3, 30m^5n^3$

**4.** $35x^3y^2, 10x^4y, 5x^5y^3$

**5.** $27p^2q^3, 45p^3q^4, 9p^4q^3$

### Factor the Greatest Common Factor from a Polynomial

*In the following exercises, factor the greatest common factor from each polynomial.*

**6.** $12x^3-10x$

**7.** $5x^3-15x^2+20x$

**8.** $8m^2-40m+16$

**9.** $24x^3-12x^2+15x$

**10.** $24y^3-18y^2-30y$

**11.** $12xy^2+18x^2y^2-30y^3$

**12.** $-5y^3+35y^2-15y$

**13.** $-4p^3q-12p^2q^2+16pq^2$

**14.** $-6a^3b-12a^2b^2+18ab^2$

**15.** $5x(x+1)+3(x+1)$

**16.** $2x(x-1)+9(x-1)$

**17.** $3b(b-2)-13(b-2)$

### Factor by Grouping

*In the following exercises, factor by grouping.*

**18.** $ab+5a+3b+15$

**19.** $pq-10p+8q-80$

**20.** $uv-18-9u+2v$

**21.** $6y^2+7y+24y+28$

**22.** $x^2-x+4x-4$

**23.** $u^2-u+6u-6$

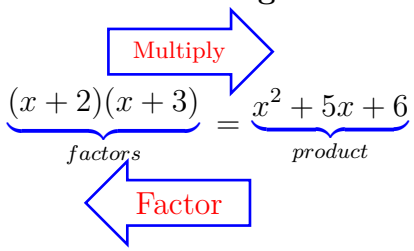
**24.** $mn-6m-4n+24$

**25.** $r^2-3r-r+3$

**26.** $2x^2-14x-5x+35$

## 4.2 Factor Trinomials

Splitting a product into factors is called **factoring**.




### 4.2.1 Factor Trinomials of the Form $x^2 + bx + c$

To figure out how we would factor a trinomial of the form $x^2 + bx + c$, such as $x^2 + 5x + 6$ and factor it to $(x + 2)(x + 3)$, lets start with two general binomials of the form $(x + m)$ and $(x + n)$.

$$\begin{aligned}(x + m)(x + n) &= x^2 + mx + nx + mn \\ &= x^2 + (m + n)x + mn \\ &= \overbrace{x^2 + (m + n)x + mn}^{x^2 + \quad b \quad x + \quad c}\end{aligned}$$

This tells us that to factor a trinomial of the form $x^2 + bx + c$, we need two factors $(x + m)$ and $(x + n)$ where the two numbers $m$ **and** $n$ ***multiply to*** $c$ ***and add to*** $b$.

**Example 4.7: How to factor a trinomial of the form $x^2 + bx + c$**

Factor: $x^2 + 11x + 24$

**Solution:**

| | | |
|---|---|---|
| Step 1: Write the factors as two binomials with first terms $x$. | Write two sets of parantheses and put $x$ as the first term. | $x^2+11x+24 = (x+\ \ )(x+\ \ )$ |
| Step 2: Find two numbers $m$ and $n$ that<br>✤ multiply to $c$, $m \cdot n = c$<br>✤ add to $b$, $m + n = b$ | Find two numbers that multiply to 24 and add to 11.<br>Factors of 24 \| Sum of factors<br>1, 24 \| 1+24=25<br>2, 12 \| 2+12=14<br>3, 8 \| 3+8=11*<br>4, 6 \| 4+6=10 | |
| Step 3: Use $m$ and $n$ as the last terms of the factors. | Use 3 and 8 as the last terms of the factors. | $x^2+11x+24 = (x+3)(x+8)$ |
| Step 4: Check by multiplying the factors. | Multiply $(x + 3)(x + 8)$. Is the product the original expression? | $(x + 3)(x + 8) =$<br>$x^2 + 8x + 3x + 24$<br>$= x^2 + 11x + 24$ |

***How To* :: How to factor a trinomial of the form $x^2 + bx + c$**

Step 1. Write the factors as two binomials with first terms $x$.
$x^2 + bx + c = (x \quad )(x \quad )$




Step 2. Find two numbers $m$ and $n$ that

- multiply to $c$, $m \cdot n = c$
- add to $b$, $m + n = b$

Step 3. Use $m$ and $n$ as the last terms of the factors. $(x + m)(x + n)$

Step 4. Check by multiplying the factors.

Now, what if the last term in the trinomial is negative? The last term is the product of the last terms in the two binomials. A negative product results from multiplying two numbers with opposite signs. You have to be very careful to choose factors to make sure you get the correct sign for the middle term, too. How do you get a negative product and a positive sum? We use one positive and one negative number. When we factor trinomials, we must have the terms written in descending order-in order from highest degree to lowest degree.

**Example 4.8:** Factor: $2x + x^2 - 48$

**Solution:**

First we put the terms in decreasing degree order. $x^2 + 2x - 48$

Factors will be two binomials with first terms $x$. $x^2 + 2x - 48 = (x \quad )(x \quad )$

Find two numbers that: multiply to -48 and add to 2.

| **Factors of $-48$** | **Sum of factors** |
|---|---|
| -1, 48 | -1 + 48 = 47 |
| -2, 24 | -2 + 24 = 20 |
| -3, 16 | -3 + 16 = 13 |
| -4, 12 | -4 + 12 = 8 |
| -6, 8 | -6 + 8 = 2* |

Use 6, 8 as the last terms of the binomials.

$$x^2 + 2x - 48 = (x - 6)(x + 8)$$

---

Sometimes youll need to factor trinomials of the form $x^2 + bxy + cy^2$ with two variables, such as $x^2 + 12xy + 36y^2$. The first term, $x^2$, is the product of the first terms of the binomial factors, $x \cdot x$. The $y^2$ in the last term means that the second terms of the binomial factors must each contain $y$. To get the coefficients $b$ and $c$, you use the same process summarized in How To Factor trinomials4.2.2.

**Example 4.9:** Factor: $r^2{-}8rs{-}9s^2$

**Solution:** We need $r$ in the first term of each binomial and $s$ in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

Note that the first terms are $r$, last terms contain $s$. $r^2{-}8rs{-}9s^2 = (r \quad s)(r \quad s)$

Find two numbers that: multiply to $-9$ and add to $-8$.

| Factors of -9 | Sum of factors |
|---|---|
| 1, -9 | $1 + (-9) = -8^*$ |
| -1, 9 | $-1 + 9 = 8$ |
| 3, -3 | $3 + (-3) = 0$ |

Use 1, $-9$ as coefficients of the last terms.

$$r^2{-}8rs{-}9s^2 = (r + s)(r - 9s)$$

### 4.2.2 Factor Trinomials of the Form $ax^2 + bx + c$ using the $ac$ Method

The $ac$ method is actually an extension of the methods you used in the last section to factor trinomials with leading coefficient one. This method is very structured (that is step-by-step), and it always works!

***How To* :: How to factor a trinomial of the form $ax^2 + bx + c$ using the '$ac$' method**

Step 1. Factor any GCF.

Step 2. Find the product $ac$.

Step 3. Find two numbers $m$ and $n$ that

How?

- multiply to $ac$, $m \cdot n = ac$
- add to $b$, $m + n = b$

Step 4. Split the middle term using $m$ and $n$. $ax^2 + mx + nx + c$

Step 5. Factor by grouping.

Step 6. Check by multiplying the factors.

**Example 4.10: How to factor a trinomial of the form $ax^2 + bx + c$**

Factor using the $ac$ method: $6x^2 + 7x + 2$

**Solution:**

| | | |
|---|---|---|
| Step 1: Factor any GCF. | Is there any greatest common factor? No! | $6x^2+7x+2$ |
| Step 2: Find the product $ac$. | $a \cdot c = 6 \cdot 2 = 12$ | $\overset{ax^2+bx+c}{6x^2+7x+2}$ |
| Step 3: Find the product $ac$.<br>✤ multiply to $ac$, $m \cdot n = ac$<br>✤ add to $b$, $m+n=b$ | Find two numbers that multiply to 12 and add to 7. Both factors must be positive.<br>$3 \cdot 4 = 12$ & $3+4=7$ | |
| Step 4: Split the middle term using $m$ and $n$.<br>$ax^2+bx+c$<br>$bx$<br>$ax^2+mx+nx+c$ | Rewrite $7x$ as $3x+4x$. It would also give the same result if we use $4x+3x$. Notice that $6x^2+3x+4x+2$ is equal to $6x^2+7x+2$. We just split the middle term to get a more useful form. | $6x^2+7x+2$<br>$\underbrace{6x^2+3x}+\underbrace{4x+2}$ |
| Step 5: Factor by grouping. | | $3x(2x+1)+2(2x+1)$<br>$(2x+1)(3x+2)$ |
| Step 6: Check by multiplying the factors. | | $(2x+1)(3x+2)$<br>$=6x^2+4x+3x+2$<br>$=6x^2+7x+2$✓ |

**Example 4.11:** Factor using the $ac$ method: $10y^2-55y+70$

**Solution:** Is there a greatest common factor? Yes. The GCF is 5. Factor it.

$10y^2-55y+70 = 5(2y^2-11y+14)$ Factor out 5.

$= 5(\overbrace{2y^2-11y+14}^{ax^2+bx+c})$ The trinomial inside the parentheses has a leading coefficient that is not 1.

$ac = 28$ Find the product $ac$.

$(-4)(-7) = 28$ Find two numbers that multiply to ac

$-4+(-7) = -11$ and add to b.

$10y^2-55y+70 = 5(2y^2-11y+14)$ Split the middle term.

$= 5(\underbrace{2y^2-7y}-\underbrace{4y+14})$

$= 5[y(2y-7)-2(2y-7)]$ Factor the trinomial by grouping.

$= 5(2y - 7)(y - 2)$

## 4.2 Section Exercises

**Factor Trinomials of the Form $x^2 + bx + c$**

*In the following exercises, factor each trinomial of the form $x^2 + bx + c$.*

**1.** $p^2 + 11p + 30$

**2.** $w^2 + 10w + 21$

**3.** $n^2 + 19n + 48$

**4.** $x^2 - 8x + 12$

**5.** $q^2 - 13q + 36$

**6.** $x^2 - 8x + 7$

**7.** $5p - 6 + p^2$

**8.** $6n - 7 + n^2$

**9.** $8 - 6x + x^2$

**10.** $x^2 - 12 - 11x$

**11.** $-11 - 10x + x^2$

**12.** $7x + x^2 + 6$

*In the following exercises, factor each trinomial of the form $x^2 + bxy + cy^2$.*

**13.** $x^2 - 2xy - 80y^2$

**14.** $p^2 - 2pq - 35q^2$

**15.** $a^2 + 5ab - 24b^2$

**16.** $r^2 + 3rs - 28s^2$

**17.** $x^2 - 13xy - 14y^2$

**18.** $u^2 - 8uv + 12v^2$

**Factor Trinomials of the Form $ax^2 + bx + c$ using the $ac$ Method**

*In the following exercises, factor using the ac method.*

**19.** $3x^2 + 5x + 2$

**20.** $2w^2 + 3w - 5$

**21.** $3x^2 + x - 10$

**22.** $5s^2 - 9s + 4$

**23.** $6y^2 + y - 15$

**24.** $5n^2 + 21n + 4$

**25.** $6n^2 - n - 2$

**26.** $4x^2 + 11x - 3$

**27.** $3u^2 + 8u + 5$

## 4.3 Factor Special Products

We have seen that some binomials and trinomials result from special products-squaring binomials and multiplying conjugates. If you learn to recognize these kinds of polynomials, you can use the special products patterns to factor them much more quickly.

### 4.3.1 Factor Perfect Square Trinomials

Some trinomials are perfect squares. They result from multiplying a binomial times itself. We squared a binomial using the Binomial Squares pattern in a previous chapter.

$$\begin{array}{ccccc} & & (a+b)^2 & & \\ & & (3x+4)^2 & & \\ a^2 & + & 2ab & + & b^2 \\ (3x)^2 & + & 2\cdot(3x)\cdot 4 & + & 4^2 \\ & & 9x^2+24x+16 & & \end{array}$$

The trinomial $9x^2 + 24x + 16$ is called a perfect square trinomial. It is the square of the binomial $3x + 4$.

If you recognize that the first and last terms are squares and the trinomial fits the perfect square trinomials pattern, you will save yourself a lot of work.

Here is the pattern-the reverse of the binomial squares pattern.

The sign of the middle term determines which pattern we will use. When the middle term is negative, we use the pattern $a^2-2ab+b^2$, which factors to $(a-b)^2$.

The steps are summarized here.

***How To* Factor Perfect Square Trinomials.**

How?

Step 1. Does the trinomial fit the pattern?

Is the first term a perfect square?

Write it as a square.

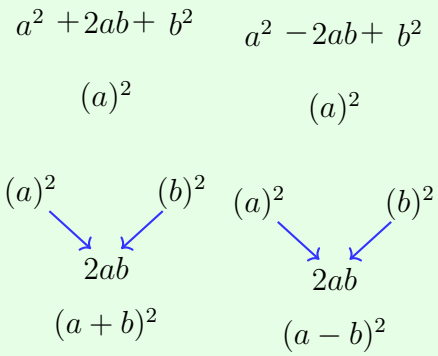
Is the last term a perfect square?

Write it as a square.

Check the middle term. Is it $2ab$ ?

Step 2. Write the square of the binomial.

Step 3. Check by multiplying the factors.



**Example 4.12: Factor:** $9x^2 + 12x + 4$

**Solution:**

| | | |
|---|---|---|
| Step 1: Does the trinomial fit the perfect square trinomial pattern, $a^2 + 2ab + b^2$?<br>✤ Is the first term a perfect square? Write it as a square $a^2$.<br>✤ Is the last term a perfect square? Write it as a square $b^2$.<br>✤ Check the middle term. Is it $2ab$? | Is $9x^2$ a perfect square? Yes - write it as $(3x)^2$.<br>Is 4 a perfect square? Yes - write it as $2^2$.<br>Is $12x$ twice the product of $3x$ and 2?<br>Does it match? Yes, so we have a perfect square trinomial. Yes - write it as $(3x)^2$. | $9x^2 + 12x + 4$<br>$(3x)^2$<br>$(3x)^2 \qquad 2^2$<br>$(3x)^2 \qquad 2^2$<br>$2(3x)2$<br>$12x$ |
| Step 2: Write the square of the binomial. | | $9x^2 + 12x + 4$<br>$a^2 + 2 \cdot a \cdot b + b^2$<br>$(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$<br>$(a+b)^2$<br>$(3x+2)^2$ |
| Step 3: Check. | | $(3x+2)^2$<br>$= (3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$<br>$= 9x^2 + 12x + 4$✓ |

The next example will be a perfect square trinomial with two variables.

**Example 4.13:** Factor: $36x^2 - 84xy + 49y^2$

**Solution:**

$36x^2 - 84xy + 49y^2$

Test each term to verify the pattern.

$a^2 - 2 \cdot a \cdot b + b^2$

$(6x)^2 - 2 \cdot (6x) \cdot (7y) + (7y)^2$

Factor.

$(6x - 7y)^2$

Remember the first step in factoring is to look for a greatest common factor.

**Example 4.14: Factor:** $100x^2y - 80xy + 16y$

**Solution:**

$100x^2y - 80xy + 16y$

Is there a GCF? Yes, $4y$, so factor it out.

$4y(25x^2 - 20x + 4)$

Is this a perfect square trinomial?
Verify the pattern.

$a^2 - 2 \cdot a \cdot b + b^2$

$4y[(5x)^2 - 2 \cdot (5x) \cdot (2) + 2^2]$

Factor.

$4y(5x - 2)^2$

### 4.3.2 Factor Differences of Squares

The other special product you saw in the previous chapter was the Product of Conjugates pattern. You used this to multiply two binomials that were conjugates. Heres an example:

$$(3x-4)(3x+4) \overset{(a-b)(a+b)}{=} (3x)^2-(4)^2 \overset{a^2-b^2}{=} 9x^2-16$$

A difference of squares factors to a product of conjugates.

**Difference of Squares Pattern**

$$a^2-b^2=(a-b)(a+b)$$

**Example 4.15:** **Factor:** $64y^2-1$

**Solution:**

$64y^2-1 = (8y)^2-1^2$ Is the binomial a difference of squares? Yes.

$= (8y-1)(8y+1)$ Factor as a product of conjugates.

As always, you should look for a common factor first whenever you have an expression to factor. Sometimes a common factor may disguise the difference of squares and you wont recognize the perfect squares until you factor the GCF. Also, to completely factor the binomial in the next example, well factor a difference of squares twice!

**Example 4.16:** Factor: $48x^4y^2-243y^2$

**Solution:**

$48x^4y^2-243y^2 = 3y^2(16x^4-81)$ Is there a GCF? Yes, $3y^2$ factor it out!

$= 3y^2\left[(4x^2)^2-(9)^2\right]$ Is the binomial a difference of squares? Yes.

$= 3y^2(4x^2-9)(4x^2+9)$ Factor as a product of conjugates.

$= 3y^2\left[(2x)^2-(3)^2\right](4x^2+9)$ Notice the first binomial is also a difference of squares!

$= 3y^2(2x-3)(2x+3)(4x^2+9)$ Factor it as the product of conjugates.

### 4.3.3 Factor sums and differences of cubes

There is another special pattern for factoring, one that we did not use when we multiplied polynomials. This is the pattern for the sum and difference of cubes.

**Sum and Difference of Cubes Pattern**

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$
$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

The two patterns look very similar, dont they? But notice the signs in the factors. The sign of the binomial factor matches the sign in the original binomial. And the sign of the middle term of the trinomial factor is the opposite of the sign in the original binomial. If you recognize the pattern of the signs, it may help you memorize the patterns.

**Example 4.17:** **Factor:** $x^3 + 64$

**Solution:**

$x^3 + 64 = (x)^3 + (4)^3$ Write the terms as cubes.

$\overset{(a+b)(a^2-ab+b^2)}{=}(x + 4)(x^2 - 4x + 4^2)$ Use the sum of cubes pattern.

$= (x + 4)(x^2 - 4x + 16)$ Simplify.

---

**Example 4.18:** Factor: $8x^3 - 27y^3$

**Solution:** This binomial is a difference. The first and last terms are perfect cubes.

$8x^3 - 27y^3 = (2x)^3 - (3y)^3$ Write the terms as cubes.

$\overset{(a-b)(a^2+ab+b^2)}{=}(2x - 3y)\left((2x)^2 + (2x)(3y) + (3y)^2\right)$ Use the difference of cubes pattern.

$= (2x - 3y)\left(4x^2 + 6xy + 9y^2\right)$ Simplify.

---

## 4.3 Section Exercises

### Factor Perfect Square Trinomials

*In the following exercises, factor completely using the perfect square trinomials pattern.*

**1.** $16y^2 + 24y + 9$

**2.** $25v^2 + 20v + 4$

**3.** $36s^2 + 84s + 49$

**4.** $100x^2 - 20x + 1$

**5.** $49x^2 + 28xy + 4y^2$

**6.** $75u^4 - 30u^3v + 3u^2v^2$

**7.** $64x^2y - 96xy + 36y$

### Factor Differences of Squares

*In the following exercises, factor completely using the difference of squares pattern, if possible.*

**8.** $25v^2 - 1$

**9.** $4 - 49x^2$

**10.** $36p^2 - 49q^2$

**11.** $x^2 - 16x + 64 - y^2$

**12.** $a^2 + 6a + 9 - 9b^2$

**13.** $6p^2q^2 - 54p^2$

### Factor Sums and Differences of Cubes

*In the following exercises, factor completely using the sums and differences of cubes pattern, if possible.*

**14.** $x^3 + 125$

**15.** $x^3 - 125$

**16.** $z^6 - 27$

**17.** $125 - 27w^3$

**18.** $7k^3 + 56$

**19.** $27y^3 + 8z^3$

**20.** $(x + 3)^3 + 8x^3$

**21.** $(y - 5)^3 - 64y^3$

**22.** $-2x^3y^2 - 16y^5$

# Chapter 5

# RATIONAL EXPRESSIONS

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**Learning outcome covered:**

1. `Simplify rational expressions and rationalize numerators or denominators.`

**Learning Objectives**

**By the end of this chapter, the students will be able to:**

- Simplify rational expressions
- Multiply and Divide rational expressions
- Find the least common denominator of rational expressions
- Add and subtract rational expressions
- Simplify a complex rational expression by writing it as division
- Simplify a complex rational expression by using the LCD

# Introduction

Rational expressions and rational equations can be useful tools for representing real life situations and for finding answers to real problems. In particular, they are quite good for describing distance-speed-time questions, and modeling multi-person work problems. In this chapter, you will work with rational expressions and perform operations on them.

## 5.1 Multiplication and Division of Rational Expressions

We previously reviewed the properties of fractions and their operations. We introduced rational numbers, which are just fractions where the numerators and denominators are integers. In this chapter, we will work with fractions whose numerators and denominators are polynomials. We call this kind of expression a **rational expression**.

**Rational Expression**

A rational expression is an expression of the form $\frac{p}{q}$, where $p$ and $q$ are polynomials and $q \neq 0$.

Here are some examples of rational expressions:

$$-\frac{25}{56} \qquad \frac{5x}{12y} \qquad \frac{4x+1}{x^2-9} \qquad \frac{4x^2+3x-1}{2x-8}$$

Notice that the first rational expression listed above, $-\frac{25}{56}$, is just a fraction. Since a constant is a polynomial with degree zero, the ratio of two constants is a rational expression, provided the denominator is not zero. We will do the same operations with rational expressions that we did with fractions. We will simplify, add, subtract, multiply, divide and use them in applications.

### 5.1.1 Simplify Rational Expressions

A fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator. Similarly, a **simplified rational expression** has no common factors, other than 1, in its numerator and denominator.

**Simplified Rational Expression**

A rational expression is considered simplified if there are no common factors in its numerator and denominator.

For example,

$\frac{x+2}{x+3}$ is simplified because there are no common factors of $x+2$ and $x+3$.

$\frac{2x}{3x}$ is not simplified because $x$ is a common factor of $2x$ and $3x$.

We use the Equivalent Fractions Property to simplify numerical fractions.We restate it here as we will also use it to simplify rational expressions.

**Equivalent Fractions Property**

If $a$, $b$, and $c$ are numbers where $b \neq 0$, $c \neq 0$,

$$\text{then } \frac{a}{b} = \frac{a \cdot c}{b \cdot c} \text{ and } \frac{a \cdot c}{b \cdot c} = \frac{a}{b}$$

Notice that in the Equivalent Fractions Property, the values that would make the denominators zero are specifically disallowed. We see $b \neq 0$, $c \neq 0$ clearly stated.

To simplify rational expressions, we first write the numerator and denominator in factored form. Then we remove the common factors using the Equivalent Fractions Property.

Be very careful as you remove common factors. Factors are multiplied to make a product. You can remove a factor from a product. You cannot remove a term from a sum.

| | | |
|---|---|---|
| $\frac{2 \cdot \cancel{3} \cdot \cancel{7}}{\cancel{3} \cdot 5 \cdot \cancel{7}}$ | $\frac{3x\cancel{(x-9)}}{5\cancel{(x-9)}}$ where $x \neq 9$ | $\frac{x+5}{x}$ where $x \neq 0$ |
| $\frac{2}{5}$ | $\frac{3x}{5}$ where $x \neq 9$ | **No common factor** |
| We remove the common factors 3 and 7. They are factors of the product. | We remove the common factors $(x-9)$. It is a factors of the product. | While there is an $x$ in both the numerator and denominator, the $x$ in the numerator is a term of a sum! |

**Example 5.1: How to simplify a rational expression.**

Simplify: $\frac{x^2+5x+6}{x^2+8x+12}$

**Solution:**

| | | |
|---|---|---|
| Step 1: Factor the numerator and denominator completely. | Factor $x^2+5x+6$ and $x^2+8x+12$. | $\frac{x^2+5x+6}{x^2+8x+12}$<br>$\frac{(x+2)(x+3)}{(x+2)(x+6)}$ |
| Step 2: Simplify by dividing out common factors. | Remove the common factor $(x+2)$ from the numerator and denominator. | $\frac{\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x+6)}$<br>$\frac{(x+3)}{(x+6)}$<br>$x \neq -2, \quad x \neq -6$ |

We now summarize the steps you should follow to simplify rational expressions.

***How To* :: Simplify a Rational Expression**




Step 1. Factor the numerator and denominator completely.

Step 2. Simplify by dividing out common factors.

Every time we write a rational expression, we should make a statement disallowing values that would make a denominator zero. However, to let us focus on the work at hand, we will omit writing it in the examples.

**Example 5.2:** Simplify: $\dfrac{3a^2-12ab+12b^2}{6a^2-24b^2}$

**Solution:**

$$\frac{3a^2-12ab+12b^2}{6a^2-24b^2} = \frac{3(a^2-4ab+4b^2)}{6(a^2-4b^2)}$$

Factor the numerator and denominator, first factoring out the GCF.

$$= \frac{3(a-2b)(a-2b)}{6(a-2b)(a+2b)}$$

$$= \frac{\cancel{3}(\cancel{a-2b})(a-2b))}{\cancel{3}\cdot 2(\cancel{a-2b})(a+2b)}$$

Remove the common factors of $a-2b$ and 3.

$$= \frac{a-2b}{2(a+2b)}$$

**Example 5.3:** Simplify: $\dfrac{x^2-4x-32}{64-x^2}$

**Solution:**

$$\frac{x^2-4x-32}{64-x^2} = \frac{(x-8)(x+4)}{(8-x)(8+x)}$$

Factor the numerator and the denominator.

$$= (-1)\frac{\cancel{(x-8)}(x+4)}{\cancel{(x-8)}(8+x)}$$

Recognize the factors that are opposites. $8-x=-(x-8)$

$$= -\frac{x+4}{x+8}$$

Simplify.

### 5.1.2 Multiply Rational Expressions

To multiply rational expressions, we multiply the numerators and multiply the denominators. Then, if there are any common factors, we remove them to simplify the result.

**Multiplication of Rational Expressions**

If $p$, $q$, $r$, and $s$ are polynomials where $q \neq 0$, $s \neq 0$, then

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

To multiply rational expressions, multiply the numerators and multiply the denominators.

Remember, throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded.

**Example 5.4:** Multiply: $\dfrac{2x}{x^2-7x+12} \cdot \dfrac{x^2-9}{6x^2}$

**Solution:**

| | | |
|---|---|---|
| Step 1: Factor each numerator and denominator completely. | Factor $x^2 - 9$ and $x^2-7x+12$. | $\dfrac{2x}{x^2-7x+12} \cdot \dfrac{x^2-9}{6x^2}$<br>$\dfrac{2x}{(x-3)(x-4)} \cdot \dfrac{(x-3)(x+3)}{6x^2}$ |
| Step 2: Multiply the numerators and denominators. | Multiply numerators and denominators. It is helpful to write the monomials first. | $\dfrac{2x(x-3)(x+3)}{6x^2(x-3)(x-4)}$ |
| Step 3: Simplify by dividing out common factors. | Divide out the common factors.<br>Leave the denominator in the factored form. | $\dfrac{\cancel{2}\cancel{x}\cancel{(x-3)}(x+3)}{\cancel{2}\cdot 3\cancel{x}^{\cancel{1}}\cancel{(x-3)}(x-4)}$<br>$\dfrac{(x+3)}{3x(x-4)}$ |

***How To* :: Multiply Rational Expressions**

How?

Step 1. Factor each numerator and denominator completely.

Step 2. Multiply the numerators and denominators.

Step 3. Simplify by dividing out common factors.

**Example 5.5:** Multiply: $\dfrac{3a^2-8a-3}{a^2-25} \cdot \dfrac{a^2+10a+25}{3a^2-14a-5}$

**Solution:**

$$\frac{3a^2-8a-3}{a^2-25} \cdot \frac{a^2+10a+25}{3a^2-14a-5} = \frac{(3a+1)(a-3)(a+5)(a+5)}{(a-5)(a+5)(3a+1)(a-5)}$$

Factor the numerators and denominators and then multiply.

$$= \frac{\cancel{(3a+1)}(a-3)\cancel{(a+5)}(a+5)}{(a-5)\cancel{(a+5)}\cancel{(3a+1)}(a-5)}$$

Simplify by dividing out common factors.

$$= \frac{(a-3)(a+5)}{(a-5)(a-5)}$$ Simplify.

$$= \frac{(a-3)(a+5)}{(a-5)^2}$$ Rewrite $(a-5)(a-5)$ using an exponent.

### 5.1.3 Divide Rational Expressions

Just like we did for numerical fractions, to divide rational expressions, we multiply the first fraction by the reciprocal of the second.

**Division of Rational Expressions**

If $p$, $q$, $r$, and $s$ are polynomials where $q \neq 0$, $r \neq 0$ and $s \neq 0$, then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$

To divide rational expressions, multiply the first fraction by the reciprocal of the second.

We now summarize the steps you should follow to simplify rational expressions.

***How To* :: Divide Rational Expressions**

How?

Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
Step 2. Factor the numerators and denominators completely.
Step 3. Multiply the numerators and denominators together.
Step 4. Simplify by dividing out common factors.

**Example 5.6: How to Divide Rational Expressions**

Divide: $\dfrac{p^3+q^3}{2p^2+2pq+2q^2} \div \dfrac{p^2-q^2}{6}$

**Solution:**

$$\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6} = \frac{p^3+q^3}{2p^2+2pq+2q^2} \cdot \frac{6}{p^2-q^2}$$ Rewrite the division as multiplication by the reciprocal.

$$= \frac{(p+q)(p^2-pq+q^2)}{2(p^2+pq+q^2)} \cdot \frac{2 \cdot 3}{(p+q)(p-q)}$$ Factor the numerators and the denominators.

$$= \frac{(p+q)(p^2-pq+q^2) \cdot 2 \cdot 3}{2(p^2+pq+q^2)(p+q)(p-q)}$$ Multiply the numerators and the denominators.

$$= \frac{\cancel{(p+q)}(p^2-pq+q^2)\cancel{2} \cdot 3}{\cancel{2}(p^2+pq+q^2)\cancel{(p+q)}(p-q)}$$ Divide out common factors.

$$= \frac{3(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$$ Simplify.

## 5.1 Section Exercises

### Simplify Rational Expressions

*In the following exercises, simplify each rational expression.*

**1.** $\dfrac{36v^3w^2}{27vw^3}$

**2.** $\dfrac{x^2+4x-5}{x^2-2x+1}$

**3.** $\dfrac{8m^3n}{12mn^2}$

**4.** $\dfrac{a^2-4}{a^2+6a-16}$

**5.** $\dfrac{x^2-y^2}{x^3-y^3}$

**6.** $\dfrac{x^3-2x^2-25x+50}{x^2-25}$

### Multiply Rational Expressions

*In the following exercises, multiply the rational expressions.*

**7.** $\dfrac{5x^2y^4}{12xy^3}\cdot\dfrac{6x^2}{20y^2}$

**8.** $\dfrac{12a^3b}{b^2}\cdot\dfrac{2ab^2}{9b^3}$

**9.** $\dfrac{5p^2}{p^2-5p-36}\cdot\dfrac{p^2-16}{10p}$

**10.** $\dfrac{2y^2-10y}{y^2+10y+25}\cdot\dfrac{y+5}{6y}$

**11.** $\dfrac{2m^2-3m-2}{2m^2+7m+3}\cdot\dfrac{3m^2-14m+15}{3m^2+17m-20}$

**12.** $\dfrac{2n^2-3n-14}{25-n^2}\cdot\dfrac{n^2-10n+25}{2n^2-13n+21}$

### Divide Rational Expressions

*In the following exercises, divide the rational expressions.*

**13.** $\dfrac{v-5}{11-v}\div\dfrac{v^2-25}{v-11}$

**14.** $\dfrac{10+w}{w-8}\div\dfrac{100-w^2}{8-w}$

**15.** $\dfrac{x^2+3x-10}{4x}\div(2x^2+20x+50)$

**16.** $\dfrac{2y^2-10yz-48z^2}{2y-1}\div(4y^2-32yz)$

*For the following exercises, perform the indicated operations.*

**17.** $\dfrac{10m^2+80m}{3m-9}\cdot\dfrac{m^2+4m-21}{m^2-9m+20}\div\dfrac{5m^2+10m}{2m-10}$

**18.** $\dfrac{4n^2+32n}{3n+2}\cdot\dfrac{3n^2-n-2}{n^2+n-30}\div\dfrac{108n^2-24n}{n+6}$

## 5.2 Addition and Subtraction of Rational Expressions

What is the first step you take when you add numerical fractions? You check if they have a common denominator. If they do, you add the numerators and place the sum over the common denominator. If they do not have a common denominator, you find one before you add.

### 5.2.1 Add and Subtract Rational Expressions with a Common Denominator

**Rational Expression Addition and Subtraction**

If $p$, $q$ and $r$ are polynomials where $r \neq 0$, then

$$\frac{p}{r} + \frac{q}{r} = \frac{p+q}{r} \text{ and } \frac{p}{r} - \frac{q}{r} = \frac{p-q}{r}$$

**Example 5.7:** Add: $\dfrac{11x+28}{x+4} + \dfrac{x^2}{x+4}$

**Solution:**

Since the denominator is $x+4$, we must exclude the value $x = -4$.

$$\frac{11x+28}{x+4} + \frac{x^2}{x+4} = \frac{11x+28+x^2}{x+4}$$ The fractions have a common denominator, so add the numerators and place the sum over the common denominator.

$$= \frac{x^2+11x+28}{x+4}$$ Write the degrees in descending order.

$$= \frac{(x+4)(x+7)}{x+4}$$ Factor the numerator.

$$= \frac{\cancel{(x+4)}(x+7)}{\cancel{x+4}}$$ Simplify by removing common factors.

$$= x+7$$ Simplify.

---

**Example 5.8:** Subtract: $\dfrac{5x^2-7x+3}{x^2-3x-18} - \dfrac{4x^2+x-9}{x^2-3x-18}$

**Solution:**

$$\frac{5x^2-7x+3}{x^2-3x-18} - \frac{4x^2+x-9}{x^2-3x-18} = \frac{5x^2-7x+3-(4x^2+x-9)}{x^2-3x-18}$$ Subtract the numerators and place the difference over the common denominator.

$$= \frac{5x^2-7x+3-4x^2-x+9}{x^2-3x-18}$$ Distribute the sign in the numerator.

$$= \frac{x^2-8x+12}{x^2-3x-18}$$ Combine like terms.

$$= \frac{(x-2)\cancel{(x-6)}}{(x+3)\cancel{(x-6)}}$$ Simplify by removing common factors.

$$= \frac{(x-2)}{(x+3)}$$ Simplify.

**Add and Subtract Rational Expressions Whose Denominators are Opposites**

Be careful with the signs as you work with the opposites when the fractions are being subtracted.

**Example 5.9:** Subtract: $\frac{m+3}{m^2-1} + \frac{4}{1-m^2}$

**Solution:**

$$\frac{m+3}{m^2-1} + \frac{4}{1-m^2} = \frac{m+3}{m^2-1} + \frac{-1(4)}{-1(1-m^2)}$$ The denominators are opposites, so multiply the second fraction by $\frac{-1}{-1}$.

$$= \frac{m+3}{m^2-1} - \frac{4}{m^2-1}$$ Simplify the second fraction.

$$= \frac{m+3-4}{m^2-1}$$ The denominators are the same. Subtract the numerators.

$$= \frac{m-1}{(m-1)(m+1)} = \frac{1}{m+1}$$ Factor the numerator and denominator.

### 5.2.2 Add and Subtract Rational Expressions with Unlike Denominators

When we add or subtract rational expressions with unlike denominators, we will need to get common denominators.

**Find the Least Common Denominator of Rational Expressions**

***How To* :: Find the Least Common Denominator of Rational Expressions.**

How?

Step 1. Factor each denominator completely.

Step 2. List the factors of each denominator. Match factors vertically when possible.

Step 3. Bring down the columns by including all factors, but do not include common factors twice.

Step 4. Write the LCD as the product of the factors.

**Example 5.10:** Find the LCD for the expressions $\frac{8}{x^2-2x-3}$, $\frac{3x}{x^2+4x+3}$.

**Solution:** Factor each denominator completely, lining up common factors.

$$x^2-2x-3 = (x+1)\ (x-3)\ \bigcirc$$

$$x^2+4x+3 = (x+1)\ \bigcirc\ (x+3)$$

Bring down the columns.

$$\text{LCD} = (x+1)((x-3)(x+3)$$

The LCD is $(x+1)(x-3)(x+3)$.

The steps used to add rational expressions are summarized here.

***How To* :: How to add Rational Expressions with Unlike Denominators**




Step 1. Determine if the expressions have a common denominator.

- Yes - goto step 2
- No - Rewrite each rational expression with the LCD
  - Find the LCD
  - Rewrite each rational expression as an equivalent rational expression with the LCD.

Step 2. Add or subtract the rational expressions.

Step 3. Simplify, if possible.

**Example 5.11:** Add: $\frac{3}{x-3}+\frac{2}{x-2}$

**Solution:**

| | | |
|---|---|---|
| Step 1: Determine if the expressions have a common denominator.<br>• Yes - goto step 2<br>• No - Rewrite each rational expression with the LCD<br>• Find the LCD<br>• Rewrite each rational expression as an equivalent rational expression with the LCD. | No<br><br>Find the LCD of $x-3$ and $x-2$<br><br>Change into equivalent rational expression with the LCD, $(x-3)(x-2)$<br><br>Keep the denominators factored | $x-3 = (x-3)\ \bigcirc$<br>$x-2 = \bigcirc\ (x-2)$<br>LCD $= (x-3)(x-2)$<br>$\frac{3}{x-3}+\frac{2}{x-2}$<br>$\frac{3(x-2)}{(x-3)(x-2)}+\frac{2(x-3)}{(x-2)(x-3)}$<br>$\frac{3x-6}{(x-3)(x-2)}+\frac{2x-6}{(x-2)(x-3)}$ |

| | | |
|---|---|---|
| Step 2: Add or subtract the rational expressions. | Add the numerators and place the sum over the common denominator. | $\frac{3x-6+2x-6}{(x-3)(x-2)}$ |
| Step 3: Simplify, if possible. | Because $5x-12$ cannot be factored, the answer is simplified. | $\frac{5x-12}{(x-3)(x-2)}$ |

**Example 5.12:** Add: $\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$

**Solution:**

Do the expressions have a common denominator? No.

Rewrite each expression with the LCD.

Find the LCD

$$x^2-2x-3 = (x+1)(x-3)$$
$$x^2+4x+3 = (x+1)(x+3)$$
$$\text{LCD} = (x+1)((x-3)(x+3)$$

$$\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$$

$$=\frac{8(x+3)}{(x+1)(x-3)(x+3)}+\frac{3x(x-3)}{(x+1)(x+3)(x-3)}$$ Rewrite each rational expression as an equivalent rational expression with the LCD.

$$=\frac{8x+24}{(x+1)(x-3)(x+3)}+\frac{3x^2-9x}{(x+1)(x+3)(x-3)}$$ Simplify the numerators.

$$=\frac{8x+24+3x^2-9x}{(x+1)(x-3)(x+3)}$$ Add the rational expressions.

$$=\frac{3x^2-x+24}{(x+1)(x-3)(x+3)}$$ Add the rational expressions.

The numerator is a prime polynomial, so there are no common factors.

**Example 5.13:** Simplify: $\frac{2u}{u-1}+\frac{1}{u}-\frac{2u-1}{u^2-u}$

**Solution:**

$$\frac{2u}{u-1}+\frac{1}{u}-\frac{2u-1}{u^2-u}=\frac{2u}{u-1}+\frac{1}{u}-\frac{2u-1}{u(u-1)}$$ Factor the denominators.

Do the expressions have a common denominator? No.

Rewrite each expression with the LCD.

Find the LCD

$$\begin{array}{lll} u-1 & = & (u-1) \\ u & = & u \\ u^2-u & = & u\ (u-1) \\ \hline \text{LCD} & = & u(u-1) \end{array}$$

$$\frac{2u}{u-1}+\frac{1}{u}-\frac{2u-1}{u^2-u}$$

$$= \frac{2u}{u-1}+\frac{1}{u}-\frac{2u-1}{u(u-1)}$$

$$= \frac{2u\cdot u}{(u-1)u}+\frac{1\cdot(u-1)}{u(u-1)}-\frac{2u-1}{u(u-1)}$$

Rewrite each rational expression as an equivalent rational expression with the LCD.

$$= \frac{2u^2}{u(u-1)}+\frac{u-1}{u(u-1)}-\frac{2u-1}{u(u-1)}$$

Simplify the numerators.

$$= \frac{2u^2+u-1-2u+1}{u(u-1)}$$

Write as one rational expression.

$$= \frac{2u^2-u}{u(u-1)}$$

Simplify.

$$= \frac{\cancel{u}(2u-1)}{\cancel{u}(u-1)}$$

Factor the numerator, and remove common factors.

$$= \frac{2u-1}{u-1}$$

Simplify.

## 5.2 Section Exercises

### Add and Subtract Rational Expressions with a Common Denominator

**1.** $\frac{7m}{2m+n}+\frac{4}{2m+n}$

**2.** $\frac{2r^2}{2r-1}+\frac{15r-8}{2r-1}$

**3.** $\frac{3s^2}{3s-2}+\frac{13s-10}{3s-2}$

**4.** $\frac{3m^2}{6m-30}-\frac{21m-30}{6m-30}$

**5.** $\frac{5r^2+7r-33}{r^2-49}-\frac{4r^2+5r+30}{r^2-49}$

### Add and Subtract Rational Expressions whose Denominators are Opposites

**6.** $\frac{10x^2+16x-7}{8x-3}+\frac{2x^2+3x-1}{3-8x}$

**7.** $\frac{a^2+3a}{a^2-9}-\frac{3a-27}{9-a^2}$

### Add and Subtract Rational Expressions with Unlike Denominators

*In the following exercises, perform the indicated operations.*

**8.** $\frac{7}{10x^2y}+\frac{4}{5xy^2}$

**9.** $\frac{1}{12a^3b^2}+\frac{5}{9a^2b^3}$

**10.** $\frac{4}{2x+5}+\frac{2}{x-1}$

**11.** $\frac{t}{t-6}-\frac{t-2}{t+6}$

**12.** $\frac{6}{m+6}-\frac{12m}{m^2-36}$

**13.** $\frac{4}{x^2-6x+5}-\frac{3}{x^2-7x+10}$

**14.** $\frac{9p-17}{p^2-4p-21}-\frac{p+1}{7-p}$

**15.** $\frac{2t-30}{t^2+6t-27}-\frac{2}{3-t}$

**16.** $\frac{5a}{a-2}+\frac{9}{a}-\frac{2a+18}{a^2-2a}$

**17.** $\frac{6d}{d-5}+\frac{1}{d+4}-\frac{7d-5}{d^2-d-20}$

## 5.3 Simplification of Complex Rational Expressions

Complex fractions are fractions in which the numerator or denominator contains a fraction. We previously simplified complex fractions like these:

$$\frac{\frac{3}{4}}{\frac{5}{8}} \qquad \frac{\frac{x}{2}}{\frac{xy}{6}}$$

In this section, we will simplify complex rational expressions, which are rational expressions with rational expressions in the numerator or denominator.

### 5.3.1 Simplify a Complex Rational Expression by Writing it as Division

**Complex Rational Expression**

A **complex rational expression** is a rational expression in which the numerator and/or the denominator contains a rational expression.

Here are a few complex rational expressions:

$$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}} \qquad \frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}+\frac{y}{x}} \qquad \frac{\frac{2}{x+6}}{\frac{4}{x-6}+\frac{4}{x^2-36}}$$

Fraction bars tell us to divide, so we rewrite $\dfrac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-7x+3}{x^2-5x+6}}$ as the division problem:

$$\frac{6x^2-7x+2}{4x-8} \div \frac{2x^2-7x+3}{x^2-5x+6}$$

Then, we multiplied the first rational expression by the reciprocal of the second, just like we do when we divide two fractions.

**Example 5.14:** Simplify the complex rational expression by writing it as division: $\dfrac{\frac{6}{x-4}}{\frac{3}{x^2-16}}$

**Solution:**

$\dfrac{\frac{6}{x-4}}{\frac{3}{x^2-16}} = \dfrac{6}{x-4} \div \dfrac{3}{x^2-16}$ Rewrite the complex fraction as division.

$= \dfrac{6}{x-4} \cdot \dfrac{x^2-16}{3}$ Rewrite as the product of first times the reciprocal of the second.

$= \dfrac{2 \cdot 3}{x-4} \cdot \dfrac{(x-4)(x+4)}{3}$ Factor.

$= \dfrac{2 \cdot 3(x-4)(x+4)}{3(x-4)}$ Multiply.

$= \dfrac{2 \cdot \cancel{3}\cancel{(x-4)}(x+4)}{\cancel{3}\cancel{(x-4)}}$ Remove common factors.

$= 2(x+4)$ Simplify.

**Example 5.15:** Simplify the complex rational expression by writing it as division: $\dfrac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$

**Solution:**

$$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}} = \frac{\frac{1\cdot 2}{3\cdot 2}+\frac{1}{6}}{\frac{1\cdot 3}{2\cdot 3}-\frac{1\cdot 2}{3\cdot 2}}$$

Find the LCD and add the fractions in the numerator. Find the LCD and subtract the fractions in the denominator.

$$= \frac{\frac{2}{6}+\frac{1}{6}}{\frac{3}{6}-\frac{2}{6}}$$

Simplify the numerator and denominator.

$$= \frac{3}{6} \div \frac{1}{6}$$

Rewrite the complex rational expression as a division problem.

$$= \frac{3}{6} \cdot \frac{6}{1} = 3$$

Multiply the first by the reciprocal of the second.

***How To* :: How to Simplify a Complex Rational Expression Using Division**

How?

Step 1. Simplify the numerator and denominator.

Step 2. Rewrite the complex rational expression as a division problem.

Step 3. Divide the expressions.

**Example 5.16:** Simplify the complex rational expression by writing it as division: $\dfrac{n-\dfrac{4n}{n+5}}{\dfrac{1}{n+5}+\dfrac{1}{n-5}}$

**Solution:**

$$\frac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}} = \frac{\frac{n(n+5)}{1(n+5)}-\frac{4n}{n+5}}{\frac{1(n-5)}{(n+5)(n-5)}+\frac{1(n+5)}{(n-5)(n+5)}}$$

Simplify the numerator and denominator. Find common denominators for the numerator and denominator.

$$= \frac{\frac{n^2+5n}{n+5}-\frac{4n}{n+5}}{\frac{n-5}{(n+5)(n-5)}+\frac{n+5}{(n-5)(n+5)}}$$

Simplify the numerators.

$$= \frac{\frac{n^2+5n-4n}{n+5}}{\frac{n-5+n+5}{(n+5)(n-5)}}$$

Subtract the rational expressions in the numerator and add in the denominator.

$$= \frac{\frac{n^2+n}{n+5}}{\frac{2n}{(n+5)(n-5)}}$$

Simplify. (We now have one rational expression over one rational expression.)

$$= \frac{n^2+n}{n+5} \div \frac{2n}{(n+5)(n-5)}$$ Rewrite as fraction division.

$$= \frac{n^2+n}{n+5} \cdot \frac{(n+5)(n-5)}{2n}$$ Multiply the first times the reciprocal of the second.

$$= \frac{n(n+1)(n+5)(n-5)}{2n(n+5)}$$ Factor any expressions if possible.

$$= \frac{\cancel{n}(n+1)\cancel{(n+5)}(n-5)}{2\cancel{n}\cancel{(n+5)}}$$ Remove common factors.

$$= \frac{(n+1)(n-5)}{2}$$ Simplify.

### 5.3.2 Simplify a Complex Rational Expression by Using the LCD

***How To* :: How to Simplify a Complex Rational Expression Using the LCD**




Step 1. Find the LCD of all fractions in the complex rational expression.
Step 2. Multiply the numerator and denominator by the LCD.
Step 3. Simplify the expression.

Lets look at the complex rational expression we simplified one way in Example 5.15. We will simplify it here by multiplying the numerator and denominator by the LCD. When we multiply by $\frac{LCD}{LCD}$ we are multiplying by 1, so the value stays the same.

**Example 5.17:** Simplify the complex rational expression by using the LCD: $\dfrac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$

**Solution:**

The LCD of all the fractions in the whole expression is 6.

$$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}} = \frac{6\cdot\left(\frac{1}{3}+\frac{1}{6}\right)}{6\cdot\left(\frac{1}{2}-\frac{1}{3}\right)}$$ Clear the fractions by multiplying the numerator and denominator by that LCD.

$$= \frac{6\cdot\frac{1}{3}+6\cdot\frac{1}{6}}{6\cdot\frac{1}{2}-6\cdot\frac{1}{3}}$$ Distribute.

$$= \frac{2+1}{3-2}$$ Simplify.

$$= \frac{3}{1} = 3$$

**Example 5.18:** Simplify the complex rational expression by using the LCD: $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$

**Solution:**

Find the LCD of all fractions in the complex rational expression.

The LCD is $xy$.

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}} = \frac{xy\cdot\left(\frac{1}{x}+\frac{1}{y}\right)}{xy\cdot\left(\frac{x}{y}-\frac{y}{x}\right)}$$ Multiply the numerator and denominator by the LCD.

$$= \frac{xy\cdot\frac{1}{x}+xy\cdot\frac{1}{y}}{xy\cdot\frac{x}{y}-xy\cdot\frac{y}{x}}$$ Simplify the expression.

$$= \frac{y+x}{x^2-y^2}$$ Simplify.

$$= \frac{x+y}{(x+y)(x-y)}$$ Factor.

$$= \frac{\cancel{x+y}}{(x-y)\cancel{(x+y)}}$$ Cancel out common factor.

$$= \frac{1}{(x-y))}$$ Simplify.

## 5.3 Section Exercises

### Simplify a Complex Rational Expression by Writing it as Division

*In the following exercises, simplify each complex rational expression by writing it as division.*

**1.** $\dfrac{\dfrac{2a}{a+4}}{\dfrac{4a^2}{a^2-16}}$

**2.** $\dfrac{\dfrac{3b}{b-5}}{\dfrac{b^2}{b^2-25}}$

**3.** $\dfrac{\dfrac{5}{c^2+5c-14}}{\dfrac{10}{c+7}}$

**4.** $\dfrac{\dfrac{8}{d^2+9d+18}}{\dfrac{12}{d+6}}$

**5.** $\dfrac{\frac{1}{2}+\frac{5}{6}}{\frac{2}{3}+\frac{7}{9}}$

**6.** $\dfrac{\frac{2}{3}-\frac{1}{9}}{\frac{3}{4}+\frac{5}{6}}$

**7.** $\dfrac{\frac{1}{p}+\frac{p}{q}}{\frac{q}{p}-\frac{1}{q}}$

**8.** $\dfrac{\frac{1}{r}+\frac{1}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

**9.** $\dfrac{\frac{m}{n}+\frac{1}{n}}{\frac{1}{n}-\frac{n}{m}}$

# Chapter 6

# SYSTEMS OF MEASUREMENT

## Contents


**6.1 U.S. System**

6.1.1 Make Unit Conversions in the U.S. System

6.1.2 Use Mixed Units of Measurement in the U.S. System

**6.2 Metric System**

6.2.1 Make Unit Conversions in the Metric System

6.2.2 Use Mixed Units of Measurement in the Metric System

**6.3 Conversion Between U.S. and Metric Systems**

**6.4 Measurements of Temperature**


**Learning outcome covered:**

**d.** `Understand measurements and conversion from one unit to another.`

**k.** `Apply knowledge of basic algebra and trigonometry in real life problems.`

**Learning Objectives**

**By the end of this chapter, the students will be able to:**

- Make unit conversions in the U.S. system
- Use mixed units of measurement in the U.S. system
- Make unit conversions in the Metric system
- Use mixed units of measurement in the Metric system
- Convert between the U.S. and the Metric systems of measurement
- Convert between Fahrenheit and Celsius temperatures

# Introduction

There are two systems of measurement commonly used around the world, namely the **Metric System** and the **U.S. System**. Most countries use the metric system. The United States uses a different system of measurement, usually called the U.S. system. In this chapter students will learn how to convert from one unit to another.

## 6.1 U.S. System

The United States uses a system of measurement, called the U.S. system. We will look at the U.S. system first. The U.S. system of measurement uses units of

- ➥ inch, foot, yard, and mile to measure length
- ➥ pound and ton to measure weight
- ➥ cup, pint, quart and gallons to measure capacity(volume)
- ➥ seconds, minutes and hours to measure time

### 6.1.1 Make Unit Conversions in the U.S. System

The equivalencies among the basic units of the U.S. system of measurement are listed in Table 1. The table also shows, in parentheses, the common abbreviations for each measurement.

| U.S. System Units | |
|---|---|
| **Length** | **Volume** |
| 1 foot (ft) = 12 inches (in)<br>1 yard (yd) = 3 feet (ft)<br>1 mile (mi) = 5280 feet (ft) | 3 teaspoons (t) = 1 tablespoon<br>16 Tablespoons (T) = 1 cup (C)<br>1 cup (C) = 8 fluid ounces (fl oz)<br>1 pint (pt) = 2 cups (C)<br>1 quart (qt) = 2 pints (pt)<br>1 gallon (gal) = 4 quarts (qt) |
| **Weight** | **Time** |
| pound (lb) = 16 ounces (oz)<br>1 ton = 2000 pounds (lb) | 1 minute (min) = 60 seconds (s)<br>1 hour (h) = 60 minutes (min)<br>1 day = 24 hours (h)<br>1 week (wk) = 7 days<br>1 year (yr) = 365 days |

**Table 1**

In many real-life applications, we need to convert between units of measurement. We will use the identity property of multiplication to do these conversions. To use the identity property of multiplication, we write 1 in a form that will help us to convert the units.

For example, we want to convert inches to feet. We know that 1 foot is equal to 12 inches,

so we can write 1 as the fraction $\frac{1 \text{ ft}}{12 \text{ in}}$. When we multiply by this fraction, we do not change the value but just change the units.

But $\frac{12 \text{ in}}{1 \text{ ft}}$ also equals 1. How do we decide whether to multiply by $\frac{1 \text{ ft}}{12 \text{ in}}$ or $\frac{12 \text{ in}}{1 \text{ ft}}$? We choose the fraction that will make the units we want to convert ***from*** divide out. For example, suppose we wanted to convert 60 inches to feet. If we choose the fraction that has inches in the denominator, we can eliminate the inches.

$$60 \cancel{\text{in}} \cdot \frac{1 \text{ ft}}{12 \cancel{\text{in}}} = 5 \text{ ft}$$

On the other hand, if we wanted to convert 5 feet to inches, we would choose the fraction that has feet in the denominator.

$$5 \cancel{\text{ft}} \cdot \frac{12 \text{ in}}{1 \cancel{\text{ft}}} = 60 \text{ in}$$

We treat the unit words like factors and divide out common units like we do common factors.

**Example 6.1:** Asma is 66 inches tall. What is her height in feet?

**Solution:**

$$\begin{aligned}
\textbf{66 inches} &= 66 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} && \text{Write 1 as a fraction relating the units given and the units needed.} \\
&= \frac{66 \text{ inches} \cdot 1 \text{ foot}}{12 \text{ inches}} && \text{Multiply.} \\
&= \frac{66 \cancel{\text{inches}} \cdot 1 \text{ foot}}{12 \cancel{\text{inches}}} && \text{Simplify the fraction.} \\
&= \frac{66 \text{ feet}}{12} \\
&= 5.5 \text{ feet}
\end{aligned}$$

Notice that when we simplified the fraction, we first divided out the inches.
Asma is 5.5 feet tall.

---

**Example 6.2:** Jumbo, an elephant at the San Diego Safari Park, weighs almost 3.2 tons. Convert her weight to pounds.

**Solution:**

$$\begin{aligned}
\textbf{3.2 tons} &= 3.2 \text{ tons} \cdot \frac{2000 \text{ lbs}}{1 \text{ ton}} && \text{Write 1 as a fraction relating tons and pounds.} \\
&= \frac{3.2 \cancel{\text{tons}} \cdot 2000 \text{ lbs}}{1 \cancel{\text{ton}}} && \text{Simplify the fraction.} \\
&= 6400 \text{ lbs}
\end{aligned}$$

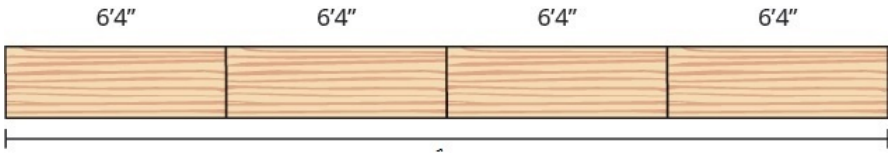
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Sometimes to convert from one unit to another, we may need to use several other units in between, so we will need to multiply several fractions.

### 6.1.2 Use Mixed Units of Measurement in the U.S. System

Performing arithmetic operations on measurements with mixed units of measures requires care. Be sure to add or subtract like units.

**Example 6.3:** Ahmed bought four planks of wood that were each 6 feet 4 inches long. If the four planks are placed end-to-end, what is the total length of the wood?




**Solution:**

| | | |
|---|---|---|
| **Total length** | = (6 feet 4 inches) × 4 | We will multiply the length of one plank by 4 to find the total length. |
| | = 24 feet + 16 inches | Multiply the inches and then the feet. |
| | = 24 feet + 1 foot 4 inches | Convert 16 inches to feet. |
| | = 25 feet + 4 inches | Add the feet. |

Ahmed bought 25 feet 4 inches of wood.

## 6.1 Section Exercises

### Make Unit Conversions in the U.S. System

*In the following exercises, convert the units.*

**1.** A park bench is 6 feet long. Convert the length to inches.

**2.** A ribbon is 18 inches long. Convert the length to feet.

**3.** Salim is 6 feet 4 inches tall. Convert his height to inches.

**4.** A football field is 160 feet wide. Convert the width to yards.

**5.** On a baseball diamond, the distance from home plate to first base is 30 yards. Convert the distance to feet.

**6.** Humaid lives 1.5 miles from school. Convert the distance to feet.

**7.** Omers dog, Beans, weighs 8 pounds. Convert his weight to ounces.

**8.** Baby Omer weighed 7 pounds 3 ounces at birth. Convert his weight to ounces.

**9.** Blue whales can weigh as much as 150 tons. Convert the weight to pounds.

### Use Mixed Units of Measurement in the U.S. System

*In the following exercises, solve and write your answer in mixed units.*

**10.** Azzan caught three fish. The weights of the fish were 2 pounds 4 ounces, 1 pound 11 ounces, and 4 pounds 14 ounces. What was the total weight of the three fishes?

**11.** Maryam bought 1 pound 6 ounces of almonds, 2 pounds 3 ounces of walnuts, and 8 ounces of cashews. What was the total weight of the nuts?

**12.** Lamees attached a 6-foot-6-inch extension cord to her computers 3-foot-8-inch power cord. What was the total length of the cords?

## 6.2 Metric System

In the metric system, units are related by powers of 10. The root words of their names reflect this relation. For example, the basic unit for measuring length is a meter. One kilometer is 1000 meters; the prefix kilo - means thousand. One centimeter is $\frac{1}{100}$ of a meter, because the prefix centi- means one one-hundredth.

### 6.2.1 Make Unit Conversions in the Metric System

The equivalencies of measurements in the metric system are shown in Table 2. The common abbreviations for each measurement are given in parentheses. We use $m$ for meter, $g$ for gram and $L$ for liter.

| Metric Measurements | | |
|---|---|---|
| **Length** | **Mass** | **Volume / Capacity** |
| 1 kilometer (km) = 1000 m<br>1 hectometer (hm) = 100 m<br>1 decameter (dam) = 10 m<br>1 decimeter (dm) = 0.1 m<br>1 centimeter (cm) = 0.01 m<br>1 millimeter (mm) = 0.001 m | 1 kilogram (kg) = 1000 g<br>1 hectogram (hg) = 100 g<br>1 decagram (dag) = 10 g<br>1 decigram (dg) = 0.1 g<br>1 centigram (cg) = 0.01 g<br>1 milligram (mg) = 0.001 g | 1 kiloliter (kL) = 1000 L<br>1 hectoliter (hL) = 100 L<br>1 decaliter (daL) = 10 L<br>1 deciliter (dL) = 0.1 L<br>1 centiliter (cL) = 0.01 L<br>1 milliliter (mL) = 0.001 L |
| 1 meter = 100 centimeters<br>1 meter = 1000 millimeters | 1 gram = 100 centigrams<br>1 gram = 1000 milligrams | 1 liter = 100 centiliters<br>1 liter = 1000 milliliters |

**Table 2**

To make conversions in the metric system, we will use the same technique we did in the U.S. system.

**Example 6.4:** Nasser ran a 10-kilometer race. How many meters did he run?

**Solution:** We will convert kilometers to meters using the Identity Property of Multiplication and the equivalencies in Table 2.

$$\begin{aligned}
\textbf{10 kilometers} &= 10 \text{ km} \times 1 && \text{Multiply the measurement to be converted by 1.}\\
&= 10 \text{ km} \times \frac{1000 \text{ m}}{1 \text{ km}} && \text{Write 1 as a fraction relating kilometers and meters.}\\
&= 10 \cancel{\text{km}} \times \frac{1000 \text{ m}}{1 \cancel{\text{km}}} && \text{Simplify.}\\
&= 10000 \text{ m} && \text{Multiply.}
\end{aligned}$$

Nasser ran 10,000 meters.

**Example 6.5:** Majids newborn baby weighed 3200 grams. How many kilograms did the baby weigh?

**Solution:** We will convert grams to kilograms.

$$\begin{aligned}
\textbf{3200 grams} &= 3200 \text{ g} \times 1 && \text{Multiply the measurement to be converted by 1.}\\
&= 3200 \text{ g} \times \frac{1 \text{ kg}}{1000 \text{ g}} && \text{Write 1 as a fraction relating kilograms and grams.}\\
&= 3200 \cancel{\text{g}} \times \frac{1 \text{ kg}}{1000 \cancel{\text{g}}} && \text{Simplify.}\\
&= 3.2 \text{ kg} && \text{Divide - moving the decimal 3 places to the left.}
\end{aligned}$$

The baby weighed 3.2 kilograms.

---

**Example 6.6:** Convert: (**a**) 350 liters to kiloliters (**b**) 4.1 liters to milliliters.

**Solution:**

**a.** We will convert liters to kiloliters. In Table 2, we see that 1 kiloliter = 1000 liters.

$$\begin{aligned}
\textbf{350 L} &= 350 \text{ L} \times 1 && \text{Multiply the measurement to be converted by 1.}\\
&= 350 \text{ L} \times \frac{1 \text{ kL}}{1000 \text{ L}} && \text{Writing 1 as a fraction relating liters to kiloliters.}\\
&= 350 \cancel{\text{L}} \times \frac{1 \text{ kL}}{1000 \cancel{\text{L}}} && \text{Simplify.}\\
&= 350 \times \frac{1 \text{ kL}}{1000}\\
&= 0.35 \text{ kL} && \text{Move the decimal 3 units to the left.}
\end{aligned}$$

**b.** We will convert liters to milliliters. In Table 2, we see that 1 liter = 1000 milliliters.

$$\begin{aligned}
\textbf{4.1 L} &= 4.1 \text{ L} \times \frac{1000 \text{ mL}}{1 \text{ L}} && \text{Multiply by 1, writing 1 as a fraction relating milliliters to liters.}\\
&= 4.1 \cancel{\text{L}} \times \frac{1000 \text{ mL}}{1 \cancel{\text{L}}} && \text{Simplify.}\\
&= 4.1 \times 1000 \text{ mL}\\
&= 4100 \text{ mL} && \text{Move the decimal 3 units to the right.}
\end{aligned}$$

---

### 6.2.2 Use Mixed Units of Measurement in the Metric System

Performing arithmetic operations on measurements with mixed units of measures in the metric system requires the same care we used in the U.S. system. But it may be easier because of the relation of the units to the powers of 10. We still must *make sure to add or subtract like units.*

**Example 6.7:** Humaid is 1.6 meters tall. His younger brother is 85 centimeters tall. How much taller is Humaid than his younger brother?

**Solution:** We will subtract the lengths in meters. Convert 85 centimeters to meters by moving the decimal 2 places to the left; 85 cm is the same as 0.85 m.

Now that both measurements are in meters, subtract to find out how much taller Humaid is than his brother.

$$\begin{array}{r} 1.60 \text{ m} \\ -0.85 \text{ m} \\ \hline 0.75 \text{ m} \end{array}$$

Humaid is 0.75 meters taller than his brother.

---

**Example 6.8:** Hibas recipe for tomato soup calls for 150 milliliters of olive oil. Hiba wants to triple the recipe. How many liters of olive oil will she need?

**Solution:** We will find the amount of olive oil in milliliters then convert to liters.

$$\begin{aligned} \textbf{Olive oil needed} &= 3 \times 150 \text{ mL} && \text{Triple 150 mL} \\ &= 450 \text{ mL} && \text{Multiply.} \\ &= 450 \text{ mL} \times \frac{1 \text{ L}}{1000 \text{ mL}} && \text{Convert to liters.} \\ &= 450 \times \frac{1 \text{ L}}{1000} && \text{Simplify.} \\ &= 0.45 \text{ L} && \text{Divide.} \end{aligned}$$

Hiba needs 0.45 liter of olive oil.

---

## 6.2 Section Exercises

### Make Unit Conversions in the Metric System

*In the following exercises, convert the units.*

**1.** Nabhan ran 5 kilometers. Convert the length to meters.

**2.** Al Jabal Al Akhdar is 2980 meters tall. Convert the height to kilometers.

**3.** Bushra is 1.55 meters tall. Convert her height to centimeters.

**4.** A multivitamin contains 1,500 milligrams of calcium. Convert this to grams.

**5.** A bottle of medicine contained 300 milliliters. Convert this to liters.

**6.** One stick of butter contains 91.6 grams of fat. Convert this to milligrams.

### Use Mixed Units of Measurement in the Metric System

*In the following exercises, solve and write your answer in mixed units.*

**7.** Ahmed mailed 5 packages that weighed 420 grams each. What was the total weight of the packages in kilograms?

**8.** Khalil is 1.6 meters tall. His sister is 95 centimeters tall. How much taller, in centimeters, is Khalil than his sister?

**9.** Murshid drinks 200 milliliters of water 8 times a day. How many liters of water does Murshid drink in a day?

## 6.3 Conversion Between U.S. and Metric Systems

Many measurements in the United States are made in metric units. To work easily in both systems, we need to be able to convert between the two systems.

| Conversion Factors Between U.S. and Metric Systems | | |
|---|---|---|
| **Length** | **Weight** | **Volume** |
| 1 in = 2.54 cm<br>1 ft = 0.305 m<br>1 yd = 0.914 m<br>1 mi = 1.61 km<br>1 m = 3.28 ft | 1 lb = 0.45 kg<br>1 oz = 28 g<br>1 kg = 2.2 lb | 1 qt = 0.95 L<br>1 fl oz = 30 mL<br>1 gallon = 3.785 L<br>1 L = 1.06 qt |

**Table 3**

We make conversions between the systems just as we do within the systemsby multiplying by unit conversion factors.

**Example 6.9:**

**a.** Mosa is 75 inches tall. Convert his height to centimeters.

**b.** Mahnad passed a football 24 yards. Convert the pass length to meters.

**c.** The height of Mount Kilimanjaro is 5,895 meters. Convert the height to feet.

**d.** The flight distance from New York City to London is 3470 miles. Convert the distance to kilometers.

**Solution:**

**a.** **75 inches**

$$= 75 \text{ in} \times \frac{2.54 \text{ cm}}{1 \text{ in}}$$

Multiply by a unit conversion factor relating inches and centimeter.

$$= 190.5 \text{ cm}$$

Simplify.

**b.** **24 yards** $= 24 \text{ yd} \times \frac{0.914 \text{ m}}{1 \text{ yd}}$ Multiply by a unit conversion factor relating yard and meter.

$= 21.936 \text{ m}$ Simplify.

**c.** **5895 meters** $= 5895 \text{ m} \times \frac{3.28 \text{ ft}}{1 \text{ m}}$ Multiply by a unit conversion factor relating meter and feet.

$= 19335.6 \text{ ft}$ Simplify.

**d.** **3470 miles** $= 3470 \text{ mi} \times \frac{1.61 \text{ km}}{1 \text{ mi}}$ Multiply by a unit conversion factor relating mile and kilometer.

$= 5586.7 \text{ km}$ Simplify.

**Example 6.10:**

**a.** Saids suitcase weighed 20 kilograms. Convert the weight to pounds.

**b.** Each American throws out an average of 1,650 pounds of garbage per year. Convert this weight to kilograms.

**Solution:**

**a.** **20 kilograms** $= 20 \text{ kg} \times \frac{2.2 \text{ lb}}{1 \text{ kg}}$ Multiply by a unit conversion factor relating kilogram and pound.

$= 44 \text{ lb}$ Simplify.

**b.** **1650 pounds** $= 1650 \text{ lb} \times \frac{0.45 \text{ kg}}{1 \text{ lb}}$ Multiply by a unit conversion factor relating kilogram and pound.

$= 742.5 \text{ kg}$ Simplify.

**Example 6.11:**

**a.** Dawood bought 8 gallons of paint. Convert the volume to liters.

**b.** Manals water bottle holds 600 mL of water. How many fluid ounces are in the bottle?

**Solution:**

**a.** **8 gallons** $= 8 \text{ gallons} \times \frac{3.785 \text{ L}}{1 \text{ gallon}}$ Multiply by a unit conversion factor relating gallon and liter.

$= 30.28 \text{ liters}$ Simplify.

**b.** **600 mL** $= 600 \text{ mL} \times \frac{1 \text{ fl oz}}{30 \text{ mL}}$ Multiply by a unit conversion factor relating fluid ounce and mL.

$= 20 \text{ fl oz}$ Simplify.

## 6.3 Section Exercises

*In the following exercises, convert between U.S. and metric units. Round to the nearest tenth.*

**1.** Majid is 61 inches tall. Convert his height to centimeters.
**2.** A college basketball court is 84 feet long. Convert this length to meters.
**3.** Rahma walked 2.5 miles. Convert this distance to kilometers.
**4.** Khalfan weighs 78 kilograms. Convert his weight to pounds.
**5.** Steves car holds 30 gallons of gas. Convert this to liters.
**6.** A box of books weighs 25 pounds. Convert this weight to kilograms.

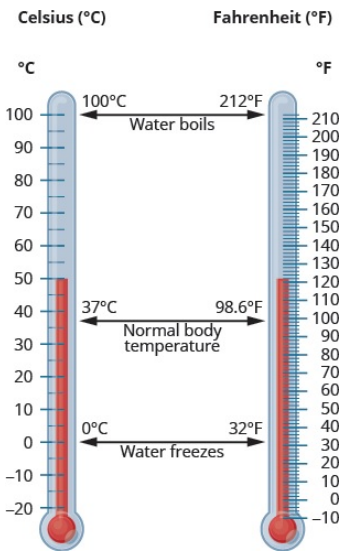
## 6.4 Measurements of Temperature

The concept of temperature has evolved from the common concepts of hot and cold. Human perception of what feels hot or cold is a relative one. Temperature is operationally defined to be what we measure with a thermometer. The U.S. and metric systems use different scales to measure temperature. The U.S. system uses degrees Fahrenheit, written °F. The metric system uses degrees Celsius, written °C. The figure shows the relationship between the two systems.

The **Celsius scale** (which replaced the slightly different centigrade scale) has the freezing point of water at 0° C and the boiling point at 100° C. Its unit is the degree Celsius ° C.

On the **Fahrenheit scale** (still the most frequently used in the United States), the freezing point of water is at 32° F and the boiling point is 180 degrees higher at 212° F. The unit of temperature on this scale is the degree Fahrenheit ° F.

The **Kelvin scale** is the temperature scale that is commonly used in science. It is an absolute temperature scale defined to have 0 K at the lowest possible temperature, called **absolute zero**. The freezing and boiling points of water are 273 K and 373 K, respectively.




If we know the temperature in one system, we can use a formula to convert it to the other system.

**Temperature Conversion**

To convert from Fahrenheit temperature, F, to Celsius temperature, C, use the formula

$$C = \frac{5}{9}(F - 32)$$

To convert from Celsius temperature, C, to Fahrenheit temperature, F, use the formula

$$F = \frac{9}{5}C + 32$$

To convert from Celsius C to Kelvin K, use the formula

$$K = C + 273$$

To convert from Kelvin K to Celsius C, use the formula

$$C = K - 273$$

To convert from Fahrenheit F to Kelvin K, use the formula

$$K = \frac{5}{9}(F - 32) + 273$$

To convert from Kelvin K to Fahrenheit F, use the formula

$$F = \frac{9}{5}(K - 273) + 32$$

**Example 6.12:** Convert 50° F into (a) degrees Celsius and (b) Kelvin.

**Solution:**

**(a)** We will substitute 50° F into the formula $C = \frac{5}{9}(F - 32)$ to find C.

$$\begin{aligned} C &= \frac{5}{9}(F - 32) && \text{Use the formula for converting } ^\circ \text{ F to } ^\circ\text{C} \\ &= \frac{5}{9}(50 - 32) && \text{Substitute 50 for F.} \\ &= \frac{5}{9}(18) && \text{Simplify in parentheses.} \\ &= 10 && \text{Multiply.} \end{aligned}$$

We have, 50°F = 10°C.

**(b)** We will substitute 50° F into the formula $K = \frac{5}{9}(F - 32) + 273$ to find K.

$$\begin{aligned} K &= \frac{5}{9}(F - 32) + 273 && \text{Use the formula for converting } ^\circ \text{ F to } ^\circ\text{C} \\ &= \frac{5}{9}(50 - 32) + 273 && \text{Substitute 50 for F.} \\ &= \frac{5}{9}(18) + 273 && \text{Simplify in parentheses.} \\ &= 10 + 273 && \text{Add.} \\ &= 283 && \text{Simplify.} \end{aligned}$$

We have, 50°F = 283 K.

*Also we can do part (b) as given below:*

$$\begin{aligned} 50^\circ F &= 10^\circ C && \text{From (a)} \\ &= 10 + 273K = 283K && \text{Convert from Celsius C to Kelvin K.} \end{aligned}$$

.....

**Example 6.13:** The weather forecast for Paris predicts a high of 20° C. Convert the temperature into (a) degrees Fahrenheit and (b) Kelvin.

**Solution:**

(a) We will substitute 20° C into the formula $F = \frac{9}{5}C + 32$ to find F.

$$\begin{aligned} F &= \frac{9}{5}C + 32 && \text{Use the formula for converting °C to °F} \\ &= \frac{9}{5}(20) + 32 && \text{Substitute 20 for C.} \\ &= 36 + 32 && \text{Simplify in parentheses.} \\ &= 68 && \text{Multiply.} \end{aligned}$$

So 20° C is equivalent to 68° F.

(b) We will substitute 20° C into the formula $K = C + 273$ to find F.

$$\begin{aligned} 20^\circ C &= 20^\circ C + 273\,K && \text{From (a)} \\ &= 293\,K && \text{Multiply.} \end{aligned}$$

.....

**Example 6.14:** The surface temperature of the Sun is about 5763 K. What is this temperature on the Fahrenheit scale? What is this on the temperature on the Celsius scale?

**Solution:**

(a) We will substitute 5763 K into the formula $F = \frac{9}{5}(K - 273) + 32$ to find F.

$$\begin{aligned} F &= \frac{9}{5}(K - 273) + 32 && \text{Use the formula for converting } F = \frac{9}{5}(K-273)+32 \\ &= \frac{9}{5}(5763 - 273) + 32 && \text{Substitute 5763 for K.} \\ &= \frac{9}{5}(5490) + 32 && \text{Simplify.} \\ &= 9 \times 1098 + 32 && \text{Simplify.} \\ &= 9914 && \text{Multiply.} \end{aligned}$$

So 5763 K is equivalent to 9914° F.

(b) We will substitute 5763 K into the formula $C = K - 273$ to find C.

$$\begin{aligned} C &= K - 273 && \text{From (a)} \end{aligned}$$

| | |
|---|---|
| $= 5763 - 273$ | Substitute 5763 for K. |
| $= 5490$ | Simplify. |

## 6.4 Section Exercises

*In the following exercises, convert the Fahrenheit temperature to degrees Celsius. Round to the nearest tenth.*

**1.** 86° F

**2.** 77° F

**3.** 104° F

**4.** 14° F

**5.** 72° F

**6.** 4° F

**7.** 0° F

**8.** 120° F

*In the following exercises, convert the Celsius temperatures to degrees Fahrenheit. Round to the nearest tenth.*

**9.** 5° C

**10.** 25° C

**11.** -10° C

**12.** -15° C

**13.** 22° C

**14.** 8° C

**15.** 43° C

**16.** 16° C

**17.** To conserve energy, room temperatures are kept at 68.0 ° F in the winter and 77.0 ° F in the summer. What are these temperatures on the Celsius scale?

**18.** What is the Fahrenheit temperature of a person with a 39° C fever?

**19.** Frost damage to most plants occurs at temperatures of 28° F or lower. What is this temperature on the Kelvin scale?

**20.** One of the hottest temperatures ever recorded on the surface of Earth was **134°F** in Death Valley, CA. What is this temperature in Celsius degrees? What is this temperature in Kelvin?

**21.** At what temperature do the Fahrenheit and Celsius scales have the same numerical value?

**22.** At what temperature do the Fahrenheit and Kelvin scales have the same numerical value?

# Chapter 7

# LINEAR EQUATIONS & INEQUALITIES

## Contents


**7.1 Linear Equations**

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7.3.2 Solving Linear Inequalities

7.3.3 Compound Inequalities

7.3.4 Applications of Linear Inequalities


**Learning outcomes covered:**

**e.** Translate worded problems into mathematical expression and model simple real life problems with equations and inequalities.

**m.** Solve linear equations, equations involving radicals, fractional expression and inequalities.

**k.** Apply knowledge of basic algebra and trigonometry in real life problems.

**Learning Objectives**

**By the end of this chapter, the students will be able to:**

- Solve linear equations using a general strategy
- Solve word problems
- Solve a formula for a specific variable
- Use formulas to solve geometry applications
- Graph inequalities on the number line
- Solve linear inequalities

# Introduction

In this chapter, you will explore linear equations and inequalities, develop a strategy for solving them, and relate them to real-world situations.

## 7.1 Linear Equations

In this section, we will study about linear equations in one or more variables and develop a strategy for solving them.

### 7.1.1 Use a General Strategy to Solve Linear Equations

Solving an equation is like discovering the answer to a puzzle. The purpose in solving an equation is to find the value or values of the variable that make each side of the equation the same so that we end up with a true statement. Any value of the variable that makes the equation true is called a **solution** to the equation. It is the answer to the puzzle!

**Solution of an equation**

A solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

✓ $x = \frac{7}{5}$ is a solution of the equation $5x + 3 = 10x - 4$. [makes a true statement $10 = 10$]

✗ $x = \frac{3}{5}$ is not a solution to the equation $5x + 3 = 10x - 4$. [makes a false statement $6 \neq 2$]

There are many types of equations that we will learn to solve. In this section we will focus on a **linear equation** in one variable.

**Linear Equation**

A **linear equation** in one variable can be written as $ax + b = 0$, where $a$ and $b$ are real numbers and $a \neq 0$.

In the next example, we will give the steps of a general strategy for solving any linear equation.

**Example 7.1: How to Solve Linear Equations Using the General Strategy**

Solve: $7(n-3)-8 = -15$

**Solution:**

| | | |
|---|---|---|
| **Step 1.** Simplify each side of the equation as much as possible. | Use the Distributive Property.<br><br>Note that each side of the equation is simplified as much as possible. | $7(n-3)-8 = -15$<br>$7n - 21 - 8 = -15$<br>$7n - 29 = -15$ |
| **Step 2.** Collect all the variable terms on one side of the equation. | Nothing to do. - All the $n$'s are on the left side. | |
| **Step 3.** Collect all the constant terms on the other side of the equation. | To get constants only on the right, add 29 to each side.<br>Simplify. | $7n - 29 + 29 = -15 + 29$<br>$7n = 14$ |
| **Step 4.** Make the coefficient of the variable term to equal to 1. | Divide each side by 7<br>Simplify. | $\frac{7n}{7} = \frac{14}{7}$<br>$n = 2$ |

***How To* :: General Strategy for Solving Linear Equations.**




Step 1. **Simplify each side of the equation as much as possible.** Use the Distributive Property to remove any parentheses. Combine like terms.

Step 2. **Collect all the variable terms on one side of the equation.** Use the Addition or Subtraction Property of Equality.

Step 3. **Collect all the constant terms on the other side of the equation.** Use the Addition or Subtraction Property of Equality.

Step 4. **Make the coefficient of the variable term to equal to 1.** Use the Multiplication or Division Property of Equality. State the solution to the equation.

Step 5. **Check the solution.** Substitute the solution into the original equation to make sure the result is a true statement.

We can solve equations by getting all the variable terms to either side of the equal sign. By collecting the variable terms on the side where the coefficient of the variable is larger, we avoid working with some negatives. This will be a good strategy when we solve inequalities later in this chapter. It also helps us prevent errors with negatives.

**Example 7.2:** Solve: $4(x-1)-2 = 5(2x+3)+6$

**Solution:**

$$4(x-1)-2 = 5(2x+3)+6$$

$$4x-4-2 = 10x+15+6 \quad \text{Distribute.}$$

$$4x-6 = 10x+21 \quad \text{Combine like terms.}$$

$$4x-6-4x = 10x+21-4x \quad \text{Subtract } 4x \text{ from each side to get the variables only on the right since } 10 > 4.$$

$$-6 = 6x+21 \quad \text{Simplify.}$$

$$-6-21 = 6x+21-21 \quad \text{Subtract 21 from each side to get the constants on left.}$$

$$-27 = 6x \quad \text{Simplify.}$$

$$\frac{-27}{6} = \frac{6x}{6} \quad \text{Divide both sides by 6.}$$

$$-\frac{9}{2} = x \quad \text{Simplify}$$

$$x = -\frac{9}{2}$$

### 7.1.2 Solve Equations with Fraction or Decimal Coefficients

We will apply the Multiplication Property of Equality and multiply both sides of an equation by the least common denominator (LCD) of all the fractions in the equation. The result of this operation will be a new equation, equivalent to the first, but without fractions.
Notice in the previous example, once we cleared the equation of fractions, the equation was like those we solved earlier in this chapter. We changed the problem to one we already knew how to solve.

***How To* :: Strategy to Solve Equations with Fraction Coefficients.**




Step 1. Find the least common denominator (LCD) of all the fractions and decimals (in fraction form) in the equation.

Step 2. Multiply both sides of the equation by that LCD. This clears the fractions.

Step 3. Solve using the General Strategy for Solving Linear Equations.

**Example 7.3:** Solve: $\frac{1}{2}y + \frac{2}{3}y - \frac{3}{4}y = 5$

**Solution:**

We want to clear the fractions by multiplying both sides of the equation by the LCD of all the fractions in the equation.

Find the LCD of all fractions in the equation $\frac{1}{2}y + \frac{2}{3}y - \frac{3}{4}y = 5$. The LCD is 12.

$$\frac{1}{2}y + \frac{2}{3}y - \frac{3}{4}y = 5$$

$$12 \cdot \left(\frac{1}{2}y + \frac{2}{3}y - \frac{3}{4}y\right) = (12)5 \qquad \text{Multiply both sides of the equation by 12.}$$

$$12 \cdot \frac{1}{2}y + 12 \cdot \frac{2}{3}y - 12 \cdot \frac{3}{4}y = (12)5 \qquad \text{Distribute.}$$

$$6y + 8y - 9y = 60 \qquad \text{Simplifynotice, no more fractions!}$$

$$5y = 60 \qquad \text{Combine like terms.}$$

$$\frac{5y}{5} = \frac{60}{5} \qquad \text{Divide both sides by 5.}$$

$$y = 12 \qquad \text{Simplify.}$$

### 7.1.3 Solve a Formula for a Specific Variable

It is often helpful to solve a formula for a specific variable. We isolate that variable on one side of the equals sign with a coefficient of one and all other variables and constants are on the other side of the equal sign.

In the sciences, we often need to change temperature from Fahrenheit to Celsius or vice versa.

**Example 7.4:** Solve the formula $C = \frac{5}{9}(F - 32)$ for $F$.

**Solution:**

$$C = \frac{5}{9}(F - 32) \qquad \text{Write the formula.}$$

$$\frac{9}{5}C = \frac{9}{5} \cdot \frac{5}{9}(F - 32) \qquad \text{Remove the fraction on the right.}$$

$$\frac{9}{5}C = F - 32 \qquad \text{Simplify.}$$

$$\frac{9}{5}C + 32 = F \qquad \text{Add 32 to both sides.}$$

$$F = \frac{9}{5}C + 32$$

Sometimes we might be given an equation that is solved for $y$ and need to solve it for $x$, or vice versa.

**Example 7.5:** Solve the formula $8x + 7y = 15$ for $y$.

**Solution:**

$$8x + 7y = 15$$

$$8x + 7y - 8x = 15 - 8x \qquad \text{Subtract } 8x \text{ from both sides to isolate the term with } y.$$

$$7y = 15 - 8x \qquad \text{Simplify.}$$

$$\frac{7y}{7} = \frac{15 - 8x}{7} \qquad \text{Divide by 7 to make the coefficient 1.}$$

$$y = \frac{15 - 8x}{7} \qquad \text{Simplify.}$$

## 7.1 Section Exercises

### Solve Equations Using the General Strategy

*In the following exercises, solve each linear equation.*

**1.** $3(10-2x)+54=0$

**2.** $-2(11-7x)+54=4$

**3.** $\frac{2}{3}(9c-3)=22$

**4.** $-15+4(2-5y)=-7(y-4)+4$

**5.** $4(p-4)-(p+7)=5(p-3)$

**6.** $4[5-8(4c-3)]=12(1-13c)-8$

### Solve Equations with Fraction or Decimal Coefficients

*In the following exercises, solve each equation with fraction coefficients.*

**7.** $\frac{1}{4}x-\frac{1}{2}=-\frac{3}{4}$

**8.** $\frac{3}{4}x-\frac{1}{2}=\frac{1}{4}$

**9.** $\frac{3x+4}{2}+1=\frac{5x+10}{8}$

### Solve a Formula for a Specific Variable

*In the following exercises, solve the formula for the specific variable.*

**10.** $8x+y=15$ for $x$

**11.** $-4x+y=-6$ for $y$

**12.** $A=\frac{1}{2}bh$ for $b$

## 7.2 Applications of Linear Equations

In this section, we will use some common geometry formulas. We will adapt our problem solving strategy to solve some word problems and geometry applications.

### 7.2.1 Word Problems

You are now well prepared and you are ready to succeed. If you take control and believe you can be successful, you will be able to master word problems.

#### Use a Problem Solving Strategy for Word Problems

Now that we can solve equations, we are ready to apply our new skills to word problems. We will develop a strategy we can use to solve any word problem successfully.

***How To* :: Solve an Application.**




Step 1. **Read** the problem. Make sure all the words and ideas are understood.

Step 2. **Identify** what we are looking for.

Step 3. **Name** what we are looking for. Choose a variable to represent that quantity.

Step 4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with the important information.

Step 5. **Solve** the equation using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

### Solve Word Problems

**Example 7.6:** The sum of seven times a number and eight is thirty-six. Find the number.

**Solution:**

| | |
|---|---|
| **Step 1: Read** the problem. | |
| **Step 2: Identify** what you are looking for. | the number |
| **Step 3: Name** what we are looking for. Choose a variable to represent that quantity. | Let $n$ = the number. |
| **Step 4: Translate.** Restate the problem in one sentence with all the important information. | The sum of seven times a number and eight $\underbrace{\quad}$ is $\underbrace{\quad}$ 36<br>$7n+8 \quad = \quad 36$ |
| **Step 5: Solve** the equation. Subtract eight from each side and simplify. Divide each side by seven and simplify. | $7n+8=36$<br>$7n=28$<br>$n=4$ |
| **Step 6: Check:** Is the sum of seven times four plus eight equal to 36? | $7n+8 \stackrel{?}{=} 36$<br>$28+8 \stackrel{?}{=} 36$<br>$36=36$ ✓ |
| **Step 7: Answer** the question. | The number is 4. |

Some number word problems ask us to find **two or more numbers**. It may be tempting to name them all with different variables, but so far, we have only solved equations with one variable. In order to avoid using more than one variable, we will define the numbers in terms of the same variable.

**Example 7.7:** The sum of two numbers is negative fifteen. One number is nine less than the other. Find the numbers.

**Solution:**

| | |
|---|---|
| **Step 1: Read** the problem. | |
| **Step 2: Identify** what you are looking for. | two numbers |
| **Step 3: Name** what you are looking for by choosing a variable to represent the first number. One number is nine less than the other. | Let $n = 1^{st}$ number.<br>$n-9 = 2^{nd}$ number |

| | |
|---|---|
| **Step 4: Translate.**<br>Write as one sentence. Translate into an equation. | The sum of two numbers is negative fifteen.<br>$\underbrace{1^{st} \text{ number} + 2^{nd} \text{ number}}$ $\underbrace{\text{was}}$ $\underbrace{\text{-15}}$<br>$n + n - 9 \quad = \quad -15$ |
| **Step 5: Solve** the equation.<br>Combine like terms.<br>Add nine to each side and simplify.<br>Simplify. | $n + n - 9 = -15$<br>$2n - 9 = -15$<br>$2n = -6$<br>$n = -3$<br>$1^{st}\ number = n = -3$<br>$2^{nd}\ number = n - 9 = -12$ |
| **Step 6: Answer** the question. | The numbers are $-3$ and $-12$. |

**Consecutive Intgers**

Some number problems involve **consecutive integers**. Consecutive integers are integers that immediately follow each other. Examples of **consecutive integers** are:

$$1, \quad 2, \quad 3, \quad 4$$
$$10, \quad 9, \quad 8, \quad 7$$

Notice that each number is one more than the number preceding it. Therefore, if we define the first integer as $n$, the next consecutive integer is $n + 1$. The one after that is one more than $n + 1$, so it is $n + 1 + 1$, which is $n + 2$.

| | |
|---|---|
| $n$ | $1^{st}$ integer |
| $n + 1$ | $2^{nd}$ consecutive integer |
| $n + 2$ | $3^{rd}$ consecutive integer etc. |

**Example 7.8:** Find three consecutive integers whose sum is $-54$.

**Solution:**

| | |
|---|---|
| **Step 1: Read** the problem. | |
| **Step 2: Identify** what you are looking for. | three consecutive integers |
| **Step 3: Name** each of the three numbers | Let $n = 1^{st}$ integer<br>$n + 1 = 2^{nd}$ consecutive integer<br>$n + 2 = 3^{rd}$ consecutive integer |
| **Step 4: Translate.**<br>Write as one sentence.<br>Translate into an equation. | The sum of the three integers is $-54$.<br>$n + n + 1 + n + 2 = -54$ |

| | |
|---|---|
| **Step 5: Solve** the equation.<br>Combine like terms.<br>Subtract three from each side.<br>Divide each side by three. | $n+n+1+n+2=-54$<br>$3n+3=-54$<br>$3n=-57$<br>$n=-19$<br>$1^{st}$ *integer*, $n=-19$<br>$2^{nd}$ *integer*, $n+1=3-19+1=-18$<br>$3^{rd}$ *integer*, $n+2==-19+2=-17$ |
| **Step 6: Check:** | $-19+(-18)+(-17) \stackrel{?}{=} -54$<br>$-54=-54$ ✓ |
| **Step 7: Answer** the question. | The three consecutive integers are $-17$, $-18$, and $-19$. |

We will expand our work to include **consecutive even integers** and **consecutive odd integers**. Consecutive even integers are even integers that immediately follow one another. Examples of consecutive even integers are: 24, 26, 28 and $-12$, $-10$, $-8$.

Notice each integer is two more than the number preceding it. If we call the first one $n$, then the next one is $n+2$. The one after that would be $n+2+2$ or $n+4$.

$n$ $1^{st}$ even integer

$n+2$ $2^{nd}$ consecutive even integer

$n+4$ $3^{rd}$ consecutive even integer etc.

Consecutive odd integers are odd integers that immediately follow one another. Consider the consecutive odd integers 63, 65, and 67.

$n$ $1^{st}$ odd integer

$n+2$ $2^{nd}$ consecutive odd integer

$n+4$ $3^{rd}$ consecutive odd integer etc.

Whether the problem asks for consecutive even numbers or odd numbers, you do not have to do anything different. The pattern is still the same - to get to the next odd or the next even integer, add two.

**Example 7.9:** Find three consecutive even integers whose sum is 120.

**Solution:**

| | |
|---|---|
| **Step 1: Read** the problem. | |
| **Step 2: Identify** what you are looking for. | three consecutive even integers |

| | |
|---|---|
| **Step 3: Name** each of the three numbers. | Let $n = 1^{st}$ even integer<br>$n + 2 = 2^{nd}$ consecutive even integer<br>$n + 4 = 3^{rd}$ consecutive even integer |
| **Step 4: Translate.**<br>Restate as one sentence.<br>Translate into an equation. | The sum of the three integers is 120.<br>$n + n + 2 + n + 4 = 120$ |
| **Step 5: Solve** the equation.<br>Combine like terms.<br>Subtract 6 from each side.<br>Divide each side by 3. | $n + n + 2 + n + 4 = 120$<br>$3n + 6 = 120$<br>$3n = 114$<br>$n = 38$<br>$1^{st}$ *even integer*, $n = 38$<br>$2^{nd}$ *even integer*, $n + 2 = 38 + 2 = 40$<br>$3^{rd}$ *even integer*, $n + 4 == 38 + 4 = 42$ |
| **Step 6: Check:** | $38 + 40 + 42 \stackrel{?}{=} 120$<br>$120 = 120$ ✓ |
| **Step 7: Answer** the question. | The three consecutive even integers are 38, 40, and 42. |

### 7.2.2 Solving Geometry Applications

***How To* :: Solve Geometry Applications.**

How?

Step 1. **Read** the problem. Make sure all the words and ideas are understood.

Step 2. **Identify** what we are looking for.

Step 3. **Name** what we are looking for. Choose a variable to represent that quantity.

Step 4. **Translate** into an equation. Write the appropriate formula for the situation. Substitute in the given information.

Step 5. **Solve** the equation using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

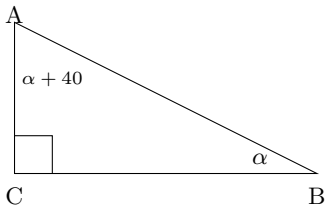
When we solve geometry applications, we often have to use some of the properties of the figures. We will review those properties as needed.

In a triangle $\triangle ABC$,

- the sum of the measures of the angles is 180°. $\angle$ A + $\angle$ B + $\angle$ C = 180.
- the perimeter of a triangle is just the distance around the triangle. We can write this as $P = a + b + c$, where $a$, $b$, and $c$ are the lengths of the sides.
- a right triangle has one 90° angle.

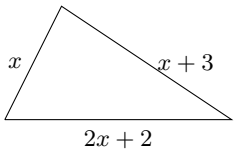
**Example 7.10:** The measure of one angle of a right triangle is 40 degrees more than the measure of the smallest angle. Find the measures of all three angles.

**Solution:**

| | |
|---|---|
| **Step 1: Read** the problem. | |
| **Step 2: Identify** what you are looking for. | the measures of all three angles |
| **Step 3: Name.** Choose a variable to represent it.<br><br>Draw the figure and label it with the given information. | Let $\alpha = 1^{st}$ angle<br>$\alpha + 40 = 2^{nd}$ angle<br>$90 = 3^{rd}$ angle(the right angle)<br>  <br> |
| **Step 4: Translate.**<br>Write the appropriate formula.<br>Substitute into the formula. | $m\angle A + m\angle B + m\angle C = 180$<br>$\alpha + (\alpha + 40) + 90 = 180$ |
| **Step 5: Solve** the equation. | $2\alpha + 130 = 180$<br>$2\alpha = 50$<br>$\alpha = 25$<br>$1^{st}$ *angle*, $\alpha = 25°$<br>$2^{nd}$ *angle*, $\alpha + 40 = 25 + 40 = 65°$<br>$3^{rd}$ *angle*, $=90°$ |
| **Step 6: Check:** | $25 + 65 + 90 \stackrel{?}{=} 180$<br>$180 = 180$ ✓ |
| **Step 7: Answer** the question. | The three angles measure 25°, 65°, and 90°. |

**Example 7.11:** One side of a triangle is three inches more than the first side. The third side is two inches more than twice the first. The perimeter is 29 inches. Find the length of the three sides of the triangle.

**Solution:**

| | |
|---|---|
| **Step 1: Read** the problem. | |
| **Step 2: Identify** what you are looking for. | the lengths of the three sides of a triangle |
| **Step 3: Name.** Choose a variable to represent the length of the first side.<br><br>Draw the figure and label it with the given information. | Let $x =$ length of $1^{st}$ side<br>$x + 3 =$ length of $2^{nd}$ side<br>$2x + 2 =$ length of $3^{rd}$ side<br>  |
| **Step 4: Translate.**<br>Write the appropriate formula.<br>Substitute into the formula. | $P = a + b + c$<br>$29 = x + (x + 3) + (2x + 2)$ |
| **Step 5: Solve** the equation. | $29 = 4x + 5$<br>$24 = 4x$<br>$6 = x$<br>*length of first side*, $x = 6$<br>*length of second side*, $x + 3 = 6 + 3 = 9$<br>*length of third side*, $2x + 2 = 2 \cdot 6 + 2 = 14$ |
| **Step 6: Check:** | $29 \stackrel{?}{=} 6 + 9 + 14$<br>$29 = 29$✓ |
| **Step 7: Answer** the question. | The lengths of the sides of the triangle are 6, 9, and 14 inches. |

## 7.2 Section Exercises

### Use a Problem Solving Strategy for Word Problems

*In the following exercises, solve using the problem solving strategy for word problems. Remember to write a complete sentence to answer each question.*

**1.** There are 16 girls in a school club. The number of girls is four more than twice the number of boys. Find the number of boys.

**2.**The difference of a number and 12 is three. Find the number.

**3.** The sum of three times a number and eight is 23. Find the number.

**4.** One number is five more than the other. Their sum is 33. Find the numbers.

**5.** The sum of two numbers is 20. One number is four less than the other. Find the numbers.

**6.** Find three consecutive integers whose sum is $-3$.

**7.** Find three consecutive integers whose sum is $-33$.

**8.** Find three consecutive even integers whose sum is 60.

**9.** Find four consecutive even integers whose sum is 92.

**10.** Find three consecutive odd integers whose sum is $-99$.

**11.** Find four consecutive odd integers whose sum is 48.

### Use Formulas to Solve Geometry Applications

*In the following exercises, solve using a geometry formula.*

**12.** The two smaller angles of a right triangle have equal measures. Find the measures of all three angles.

**13.** The angles in a triangle are such that one angle is twice the smallest angle, while the third angle is three times as large as the smallest angle. Find the measures of all three angles.

**14.** One side of a triangle is seven inches more than the first side. The third side is four inches less than three times the first. The perimeter is 28 inches. Find the length of the three sides of the triangle.

**15.** One side of a triangle is three feet less than the first side. The third side is five feet less than twice the first. The perimeter is 20 feet. Find the length of the three sides of the triangle.

## 7.3 Linear Inequalities

### 7.3.1 Graph Inequalities on the Number Line

What about the solution of an inequality? What number would make the inequality $x > 3$ true? Are you thinking, $x$ could be 4? Thats correct, but $x$ could be 5 too, or 20, or even 3.001. Any number greater than 3 is a solution to the inequality $x > 3$.

We show the solutions to the inequality $x > 3$ on the number line by shading in all the numbers to the right of 3, to show that all numbers greater than 3 are solutions. Because the number 3 itself is not a solution, we put a circle (hole) at 3.

We can also represent inequalities using **interval notation**. There is no upper end to the solution to this inequality. In interval notation, we express $x > 3$ as $(3, \infty)$. The symbol $\infty$ is read as **infinity**. It is not an actual number. Figure 7.1 shows both the number line and the interval notation.

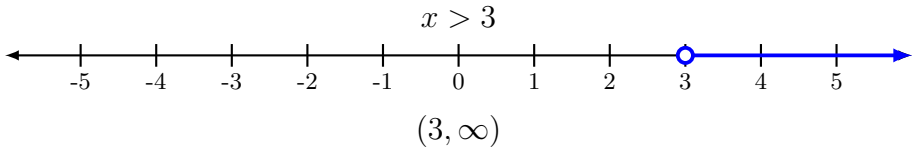



**Figure 7.1:** The inequality $x > 3$ is graphed on this number line and written in interval notation.

We use the left parenthesis symbol, (, to show that the endpoint of the inequality is not included. The left bracket symbol, [, shows that the endpoint is included.
The inequality $x \leq 1$ means all numbers less than or equal to 1. Here we need to show that one is a solution, too. We do that by putting a bracket at $x = 1$. We then shade in all the numbers to the left of one, to show that all numbers less than one are solutions. See Figure 7.2.

There is no lower end to those numbers. We write $x \leq 1$ in interval notation as $(-\infty, 1]$. The symbol $-\infty$ is read as **negative infinity**. Figure 7.2 shows both the number line and interval notation.

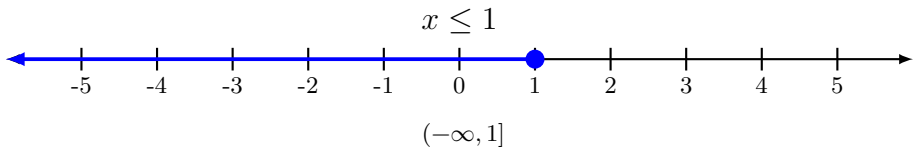
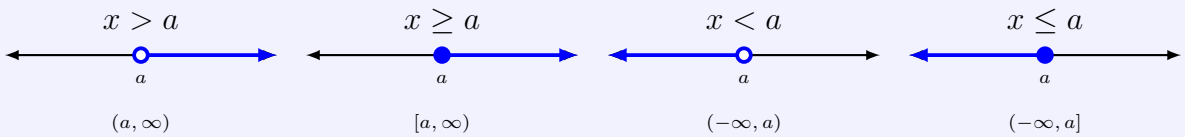



**Figure 7.2:** The inequality $x \leq 1$ is graphed on this number line and written in interval notation.

**Inequalities, Number Lines, and Interval Notation**



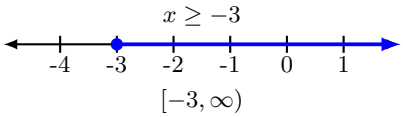
**Example 7.12:** **Graph on the number line:**

**a.** $x \geq -3$ **b.** $x < 2.5$ **c.** $x \leq -\frac{1}{2}$

**Solution:**

**a.** $x \geq -3$

Shade to the right of 3, and put a bracket at 3.

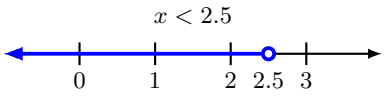



Write in interval notation.

$[-3, \infty)$

**b.** $x < 2.5$

Shade to the left of 2.5 and put a parenthesis at 2.5.

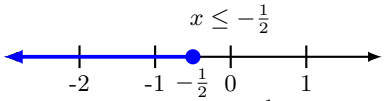



Write in interval notation.

$(-\infty, 2.5)$

**c.** $x \leq -\frac{1}{2}$

Shade to the left of $-\frac{1}{2}$, and put a bracket at $-\frac{1}{2}$.




Write in interval notation.

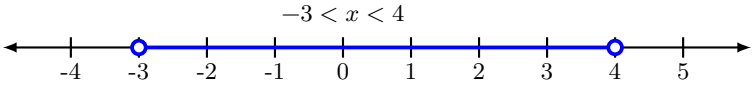
$(-\infty, -\frac{1}{2}]$

**Example 7.13:** **Graph on the number line:**

**a.** $-3 < x < 4$ **b.** $-6 \leq x < -1$ **c.** $0 \leq x \leq 2.5$

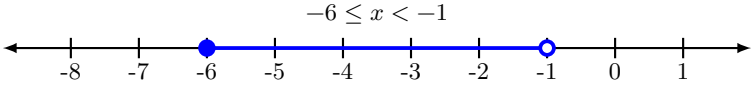
**Solution:**

**a.** $-3 < x < 4$



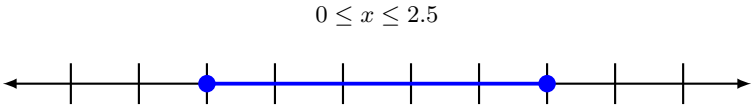

$(-3, 4)$

**b.** $-6 \leq x < -1$




$[-6, -1)$

**c.** $0 \leq x \leq 2.5$




$[0, 2.5]$

### 7.3.2 Solving Linear Inequalities

A linear inequality is much like a linear equation-but the equal sign is replaced with an inequality sign.

**Linear Inequality**

A linear inequality in one variable is an inequality that can be written in one of the following forms where $a$, $b$, and $c$ are real numbers and $a \neq 0$:

$$ax + b < c, \qquad ax + b \leq c, \qquad ax + b > c, \qquad \text{or } ax + b \geq c$$

**Addition and Subtraction Property of Inequality**

For any numbers $a$, $b$, and $c$, if $a < b$, then

$$\text{a} + \text{c} < \text{b} + \text{c} \qquad \text{a} \;\; \text{c} < \text{b} \;\; \text{c}$$

We can add or subtract the same quantity from both sides of an inequality and still keep the inequality.

Does the inequality stay the same when we divide or multiply by a negative number?

When we divide or multiply an inequality by a positive number, the inequality sign stays the same. When we divide or multiply an inequality by a negative number, the inequality sign reverses.

**Multiplication and Division Property of Inequality**

For any numbers $a$, $b$, and $c$,

**multiply or divide by a positive**

if $a < b$ and $c > 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

if $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

**multiply or divide by a negative**

if $a < b$ and $c < 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

if $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

**Example 7.14:** Solve each inequality. Graph the solution on the number line, and write the solution in interval notation:

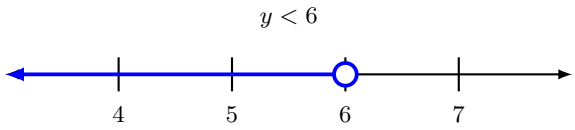
**a.** $9y < 54$ **b.** $-7x \leq -70$

**Solution:**

**a.** $9y < 54$

| | |
|---|---|
| $\frac{9y}{9} < \frac{54}{9}$ | Divide both sides of the inequality by 9; since 9 is positive, the inequality stays the same. |
| $y < 6$ | Simplify. |

Graph the solution on the number line.



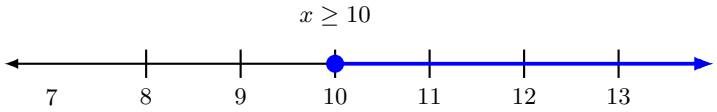

Write the solution in interval notation. $(-\infty, 6)$

---

**b.** $-7x \leq -70$

| | |
|---|---|
| $\frac{-7x}{-7} \geq \frac{-70}{-7}$ | Divide both sides of the inequality by $-7$. Since $-7$ is a negative, the inequality reverses. |
| $x \geq 10$ | Simplify. |

Graph the solution on the number line.




Write the solution in interval notation. $[10, -\infty)$

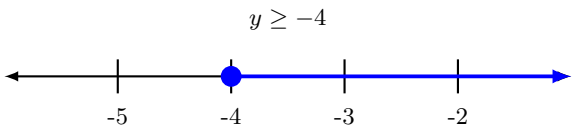
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**Example 7.15:** Solve the inequality $6y \leq 11y + 20$, graph the solution on the number line, and write the solution in interval notation.

**Solution:** $6y \leq 11y + 20$

| | |
|---|---|
| $6y - 11y \leq 11y - 11y + 20$ | Subtract $11y$ from both sides to collect the variables on the left. |
| $-5y \leq 20$ | Simplify. |
| $\frac{-5y}{-5} \geq \frac{20}{-5}$ | Divide both sides of the inequality by 5, and reverse the inequality. |
| $y \geq -4$ | Simplify. |

Graph the solution on the number line.




Write the solution in interval notation. $[-4, \infty)$

---

*When solving inequalities, it is usually easiest to collect the variables on the side where the coefficient of the variable is largest. This eliminates negative coefficients and so we dont have to multiply or divide by a negativewhich means we dont have to remember to reverse the inequality sign.*

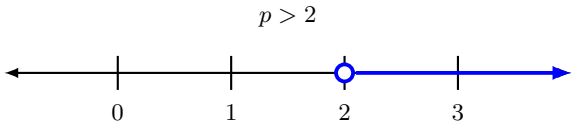
**Example 7.16:** Solve the inequality $8p + 3(p-12) > 7p-28$, graph the solution on the number line, and write the solution in interval notation.

**Solution:**

$8p + 3(p-12) > 7p-28$

| | |
|---|---|
| $8p + 3p-36 > 7p-28$ | Simplify each side as much as possible. Distribute |
| $11p-36 > 7p-28$ | Combine like terms. |
| $11p-36-7p > 7p-28-7p$ | Subtract $7p$ from both sides to collect the variables on the left, since $11 > 7$.. |
| $4p-36 > -28$ | Simplify. |
| $4p-36+36 > -28+36$ | Add 36 to both sides to collect the constants on the right. |
| $4p > 8$ | Simplify. |
| $\frac{4p}{4} > \frac{8}{4}$ | Divide both sides of the inequality by 4; the inequality stays the same.. |
| $p > 2$ | Simplify. |

Graph the solution on the number line.




Write the solution in interval notation. $(2, \infty)$

### 7.3.3 Compound Inequalities

A compound inequality is made up of two inequalities connected by the word and or the word or. For example, the following are compound inequalities.

$$x + 3 > -4 \text{ and } 4x-5 \leq 3$$
$$2(y + 1) < 0 \text{ or } y-5 \geq -2$$

Sometimes we have a compound inequality that can be written more concisely. For example, $a < x$ and $x < b$ can be written simply as $a < x < b$ and then we call it a **double inequality**. These two forms are equivalent.

To solve a double inequality we perform the same operation on all three parts of the double inequality with the goal of isolating the variable in the center.

**Example 7.17:** Solve $-4 \leq 3x-7 < 8$. Graph the solution and write the solution in interval notation.

**Solution:**

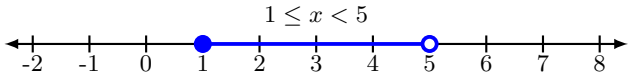
| | | | | | |
|---|---|---|---|---|---|
| $-4$ | $\leq$ | $3x-7$ | $<$ | $8$ | |
| $-4+7$ | $\leq$ | $3x$ | $<$ | $8+7$ | Add 7 to all three parts. |
| $3$ | $\leq$ | $3x$ | $<$ | $15$ | Simplify. |

$$\begin{array}{ccccc} \frac{3}{3} & \leq & \frac{3x}{3} & < & \frac{15}{3} \\ 1 & \leq & x & < & 5 \end{array}$$

Divide each part by three.

Simplify.

Graph the solution:




Write the solution in interval notation: $[1, 5)$

### 7.3.4 Applications of Linear Inequalities

#### Translate to an Inequality and Solve - Word Problems

To translate English sentences into inequalities, we need to recognize the phrases that indicate the inequality. Some words are easy, like more than and less than. But others are not as obvious. The following table shows some common phrases that indicate inequalities.

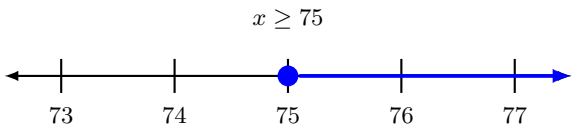
| $>$ | $\geq$ | $<$ | $\leq$ |
|---|---|---|---|
| is greater than | is greater than or equal to | is less than | is less than or equal to |
| is more than | is at least | is smaller than | is at most |
| is larger than | is no less than | has fewer than | is no more than |
| exceeds | is the minimum | is lower than | is the maximum |

**Example 7.18:** Translate and solve. Then graph the solution on the number line, and write the solution in interval notation. Twenty seven less than $x$ is at least 48.

**Solution:** Twenty seven less than $x$ is **at least** 48.

$$\begin{array}{rll} x - 27 \geq 48 & & \text{Translate} \\ x \geq 48 + 27 & & \\ x \geq 75 & & \end{array}$$

Graph on the number line:




Write the solution in interval notation: $[75, \infty)$

#### Solve Applications with Linear Inequalities

Many real-life situations require us to solve inequalities. Sometimes an application requires the solution to be a whole number, but the algebraic solution to the inequality is not a whole number. In that case, we must round up or round down the algebraic solution to a whole number. The context of the application will determine whether we round up or down.

**How to solve application problems?**

Step 1. **Read** the problem. Make sure all the words and ideas are understood.
Step 2. **Identify** what we are looking for.
Step 3. **Name** what we are looking for. Choose a variable to represent that quantity.
Step 4. **Translate**. Write a sentence that gives the information to find it. Translate into an inequality.
Step 5. **Solve** the inequality.
Step 6. **Check** the answer in the problem and make sure it makes sense.
Step 7. **Write** a sentence that answers the question.

**Example 7.19:** Alia has \$25 to spend on juice boxes for her sons preschool picnic. Each pack of juice boxes costs \$2. What is the maximum number of packs she can buy?

**Solution:** We are looking for the maximum number of packs Alia can buy.
Let $n =$ the number of packs.
Given that: \$2 times the number of packs **is less than or equal to** \$25.

$$2n \leq 25 \qquad \text{Translate into an inequality.}$$
$$n \leq \frac{25}{2}$$
$$n \leq 12.5$$
$$n \leq 12 \qquad \text{But } n \text{ must be a whole number of packs, so round to 12.}$$

Alia can buy a maximum of 12 packs.

---

**Example 7.20:** Mohammed is planning a six-day summer vacation trip. He has omr 800 in savings, and he earns omr 10 by selling one painting. The trip will cost him omr 510 for airfare, omr 700 for food and sightseeing, and omr 200 for the hotel. How many paintings must he sell to have enough money to pay for the trip?

**Solution:** We are looking for the number of painting Mohammed sells
Let $n =$ the number of paintings
The expenses must be less than or equal to the income. **The cost of airfare plus the cost of food and sightseeing plus the hotel bill must be less than or equal to the savings plus the amount earned by selling paintings.**

$$510 + 700 + 200 \leq 800 + 10n \qquad \text{Translate into an inequality.}$$
$$1410 \leq 800 + 10n$$
$$610 \leq 10n$$
$$\frac{610}{10} \leq n$$
$$n \geq 61$$
$$n \geq 61$$

Mohammed must sell at least 61 paintings.

## 7.3 Section Exercises

### Graph Inequalities on the Number Line

*In the following exercises, graph each inequality on the number line and write in interval notation.*

**1.**

**a.** $x \leq 5$

**b.** $x \geq -1.5$

**2.**

**a.** $-5 < x < 2$

**b.** $-3 \leq x < -1$

**3.**

**a.** $-2 < x < 0$

**b.** $-5 \leq x < -3$

### Solve Linear Inequalities

*In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.*

**4.** $8x > 72$

**5.** $6y < 48$

**6.** $20 \geq \frac{2}{5}x$

**7.** $-5k \leq 20$

**8.** $3x \geq -15$

**9.** $12 \leq -4y$

*In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.*

**10.** $4v \geq 9v - 40$

**11.** $12x + 3(x + 7) > 10x - 24$

**12.** $8m - 2(3 - m) \geq 2(m + 7) + 3m$

**13.** $9y + 5(y + 3) < 4y - 35$

**14.** $6h - 4(h - 1) \leq 7h - 11$

**15.** $4k - (k - 2) \geq 7k - 26$

**16.** $\frac{2}{3}b - \frac{3}{4}b < \frac{5}{12}b - \frac{1}{2}$

**17.** $6n - 12(3 - n) \leq 9(n - 4) + 9n$

**18.** $18q - 4(10 - 3q) < 5(6q - 8)$

**19.** $-\frac{9}{4}x \geq -\frac{5}{12}$

### Solve Compound Inequalities

*In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.*

**20.** $-5 \leq 4x - 1 < 7$

**21.** $5 < 4x + 1 < 9$

**22.** $-6 \leq 4x - 2 < -2$

**23.** $-1 < 3x + 2 < 8$

### Translate to an Inequality and Solve

*In the following exercises, translate and solve. Then graph the solution on the number line and write the solution in interval notation.*

**24.** Three more than $h$ is no less than 25.

**25.** Six more than $k$ exceeds 25.

**26.** Fifteen less than $a$ is at least 7.

**27.** Nineteen less than $b$ is at most 22.

### Solve Applications with Linear Inequalities

*In the following exercises, solve.*

**28.** Ryan charges his neighbors $20 to wash their car. How many cars must he wash next summer if his goal is to earn at least $1500?

**29.** Maryam got a OMR 20 gift card for the coffee shop. Her favorite iced drink costs OMR 3. What is the maximum number of drinks she can buy with the gift card?

**30.** Ahmed is a personal chef. He charges OMR 4 per meal. His monthly expenses are OMR 400. How many meals must he sell in order to make a profit of at least OMR 600?

# Chapter 8

# QUADRATIC EQUATIONS

## Contents


**8.1 Quadratic Formula**

8.1.1 Solving Quadratic Equation using Quadratic Formula

8.1.2 Use the Discriminant to predict the number of solutions of a Quadratic Equation

**8.2 Applications of Quadratic Equations**


**Learning outcome covered:**

**o.** `Use the quadratic formula to find roots of a second-degree polynomial.`

**k.** `Apply knowledge of basic algebra and trigonometry in real life problems.`

**Learning Objectives**

**At the end of this chapter, the students will be able to:**

- Solve quadratic equations using the quadratic formula
- Use the discriminant to predict the nature of solutions of a quadratic equation
- Solve applications modeled by Quadratic Equations

## Introduction

Quadratic equations are equations of the form $ax^2 + bx + c = 0$ , where $a \neq 0$. They differ from linear equations by including a term with the variable raised to the second power. We use different methods to solve quadratic equations than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

In this chapter, we solve the quadratic equations using quadratic formula and study the nature of the solutions. Also, we solve some applications modeled by quadratic equation. Quadratic equation can also be solved by factorization.

## 8.1 Quadratic Formula

### 8.1.1 Solving Quadratic Equation using Quadratic Formula

**Quadratic Formula**

The solutions to a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To use the Quadratic Formula, we substitute the values of $a$, $b$, and $c$ into the expression on the right side of the formula. Then, we do all the math to simplify the expression. The result gives the solution(s) to the quadratic equation. **How to solve a Quadratic Equation using the Quadratic Formula**

***How To* :: Solve a Quadratic Equation using the Quadratic Formula.**

How?

Step 1. Write the Quadratic Formula in standard form. Identify the $a$, $b$ and $c$ values.

Step 2. Write the Quadratic Formula. Then substitute the values of $a$, $b$ and $c$.

Step 3. Simplify.

Step 4. Check the solutions.

**Example 8.1:** Solve $2x^2 + 9x - 5 = 0$ by using the Quadratic Formula.

**Solution:**

| | | |
|---|---|---|
| **Step 1:** Write the Quadratic Formula in standard form. Identify the $a$, $b$ and $c$ values. | The equation is in standard form. | $ax^2 + bx + c = 0$<br>$2x^2 + 9x - 5 = 0$<br>$a = 2$, $b = 9$, $c = -5$ |

| | | |
|---|---|---|
| **Step 2:** Write the Quadratic Formula. Then substitute the values of $a$, $b$ and $c$. | Substitute $a = 2$, $b = 9$ and $c = -5$. | $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$<br>$x = \dfrac{-9 \pm \sqrt{9^2 - 4 \cdot 2 \cdot (-5)}}{2 \cdot 2}$ |
| **Step 3:** Simplify the fraction and solve for $x$. | | $x = \dfrac{-9 \pm \sqrt{81 + 40}}{4}$<br>$x = \dfrac{-9 \pm \sqrt{121}}{4}$<br>$x = \dfrac{-9 \pm 11}{4}$<br>$x = \dfrac{-9 + 11}{4}, \quad x = \dfrac{-9 - 11}{4}$<br>$x = \dfrac{2}{4}, \quad x = \dfrac{-20}{4}$<br>$x = \dfrac{1}{2}, \quad x = -5$ |

**Example 8.2:** Solve $x^2 - 6x + 5 = 0$ by using the Quadratic Formula.

**Solution:** This equation is in standard form.

$$\begin{gathered} {\scriptstyle ax^2 + bx + c = 0} \\ x^2 - 6x + 5 = 0 \\ a = 1, \quad b = -6, \, c = 5 \end{gathered}$$

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Write the Quadratic Formula.} \\
x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} && \text{Substitute the values of } a, \, b, \, c. \\
x &= \frac{6 \pm \sqrt{36 - 20}}{2} \\
x &= \frac{6 \pm \sqrt{16}}{2} && \text{Simplify.} \\
x &= \frac{6 \pm 4}{2} \\
x &= \frac{6 + 4}{2}, \quad x = \frac{6 - 4}{2} && \text{Rewrite to show two solutions.} \\
x &= \frac{10}{2}, \quad x = \frac{2}{2} \\
x &= 5, \quad x = 1
\end{aligned}$$

**Example 8.3:** Solve $2x^2 + 10x + 11 = 0$ by using the Quadratic Formula.

**Solution:**

$$\begin{gathered} {\scriptstyle ax^2+\ bx+\ \ c\ =\ 0} \\ 2x^2 + 10x + 11 = 0 \\ a = 2, \quad b = 10, \, c = 11 \end{gathered}$$

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Write the Quadratic Formula.} \\
x &= \frac{-10 \pm \sqrt{10^2 - 4 \cdot 2 \cdot 11}}{2 \cdot 2} && \text{Substitute the values of } a, \, b, \, c. \\
x &= \frac{-10 \pm \sqrt{100 - 88}}{4} \\
x &= \frac{-10 \pm \sqrt{12}}{4} \\
x &= \frac{-10 \pm 2\sqrt{3}}{4} && \text{Simplify the radical.} \\
x &= \frac{2(-5 \pm \sqrt{3})}{4} && \text{Factor out the common factor in the numerator.} \\
x &= \frac{-5 \pm \sqrt{3}}{2} && \text{Remove the common factors.} \\
x &= \frac{-5 + \sqrt{3}}{2}, \quad x = \frac{-5 - \sqrt{3}}{2} && \text{Rewrite to show two solutions.}
\end{aligned}$$

**Example 8.4:** Solve $3p^2 + 2p + 9 = 0$ by using the Quadratic Formula.

**Solution:**

$$\begin{gathered} {\scriptstyle ax^2+\ bx+\ \ c\ =\ 0} \\ 3p^2 + 2p + 9 = 0 \\ a = 3, \quad b = 2, \, c = 9 \end{gathered}$$

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Write the Quadratic Formula.} \\
x &= \frac{-2 \pm \sqrt{2^2 - 4 \cdot 3 \cdot 9}}{2 \cdot 3} && \text{Substitute the values of } a, \, b, \, c. \\
x &= \frac{-2 \pm \sqrt{4 - 108}}{6} && \text{Simplify.} \\
x &= \frac{-2 \pm \sqrt{-104}}{6} && \text{Simplify the radical.}
\end{aligned}$$

`We cannot take the square root of a negative number. There is no real solution.`

The quadratic equations we have solved so far in this section were all written in standard form, $ax^2 + bx + c = 0$. Sometimes, we will need to do some algebra to get the equation into standard form before we can use the Quadratic Formula.

**Example 8.5:** Solve $4x^2 - 20x = -25$ by using the Quadratic Formula.

**Solution:**

Add 25 to get the equation in standard form.

$$\underset{ax^2 + bx + c = 0}{4x^2 - 20x + 25 = 0}$$

$$a = 4, \quad b = -20, \; c = 25$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Write the Quadratic Formula.}$$

$$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4 \cdot 4 \cdot 25}}{2 \cdot 4} \qquad \text{Substitute the values of } a, \; b, \; c.$$

$$x = \frac{20 \pm \sqrt{400 - 400}}{8} \qquad \text{Simplify.}$$

$$x = \frac{20 \pm \sqrt{0}}{8} \qquad \text{Simplify inside radical.}$$

$$x = \frac{20 \pm 0}{8} \qquad \text{Simplify the radical.}$$

$$x = \frac{20 + 0}{8}, \quad x = \frac{20 - 0}{8} \qquad \text{Rewrite to show two solutions.}$$

$$x = \frac{5}{2}, \quad x = \frac{5}{2} \qquad \text{Simplify the fraction.}$$

`Did you recognize that` $4x^2 - 20x + 25 = 0$ `is a perfect square?`

### 8.1.2 Use the Discriminant to predict the number of solutions of a Quadratic Equation

When we solved the quadratic equations in the previous examples, sometimes we got two distinct real solutions, sometimes two equal solution, sometimes no real solution. Is there a way to predict the nature and the number of solutions to a quadratic equation without actually solving the equation?

Yes, the quantity inside the radical of the Quadratic Formula makes it easy for us to determine the nature as well as the number of solutions. This quantity is called the discriminant.

**Discriminant**

In the Quadratic Formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the quantity $b^2 - 4ac$ is called the **discriminant**.

***How To* :: Nature of the solutions of Quadratic equation.**




For a quadratic equation of the form $ax^2 + bx + c = 0$ , $a \neq 0$,

- ✠ if $b^2{-}4ac > 0$ , the equation has two real distinct solutions.
- ✠ if $b^2{-}4ac = 0$ , the equation has two equal solutions.
- ✠ if $b^2{-}4ac < 0$ , the equation has no real solution.

**Example 8.6:** Determine the number of solutions to each quadratic equation:

**a.** $2v^2{-}3v + 6 = 0$ **b.** $3x^2 + 7x{-}9 = 0$

**Solution:**

**a.** The equation is in standard form, identify $a$, $b$, $c$. $2v^2{-}3v + 6 = 0$; $a = 2,\ b = -3,\ c = 6$

Write the discriminant. $b^2{-}4ac$

Substitute the values of $a$, $b$, $c$. $(-3)^2{-}4 \cdot 2 \cdot 6$

Simplify. $9 - 48$; $-39$

*Because the discriminant is negative, there is* ***no real solution*** *to the equation.*

**b.** The equation is in standard form, identify $a$, $b$, $c$. $3x^2 + 7x{-}9 = 0$; $a = 3,\ b = 7,\ c = -9$

Write the discriminant. $b^2{-}4ac$

Substitute the values of $a$, $b$, $c$. $(7)^2{-}4 \cdot 3 \cdot (-9)$

Simplify. $49 + 108$; $157$

*Because the discriminant is positive, there are two distinct real solutions to the equation.*

**Example 8.7:** Determine the number of solutions to each quadratic equation:

**a.** $5n^2 + n + 4 = 0$ **b.** $9y^2 + 1 = 6y$

**Solution:**

**a.** The equation is in standard form $5n^2 + n + 4 = 0$. $a = 5,\ b = 1,\ c = 4$.

$$\begin{aligned} \textit{Discreminant} &= b^2{-}4ac \\ &= (1)^2{-}4 \cdot 5 \cdot 4 = 1 - 80 \\ &= -79 \end{aligned}$$

*Because the discriminant is negative, there is no real solution to the equation.*

**b.** Write the equation is in standard form $9y^2-6y+1=0$. $a=9$, $b=-6$, $c=1$.

$$\begin{aligned} Discreminant &= b^2-4ac \\ &= (-6)^2-4\cdot 9\cdot 1 = 36-36 \\ &= 0 \end{aligned}$$

*Because the discriminant is 0, the solutions are equal.*

## 8.1 Section Exercises

### Solve Quadratic Equations Using the Quadratic Formula

*In the following exercises, solve by using the Quadratic Formula.*

**1.** $4m^2+m-3=0$ **2.** $4n^2-9n+5=0$ **3.** $2p^2-7p+3=0$

**4.** $3q^2+8q-3=0$ **5.** $p^2+7p+12=0$ **6.** $q^2+3q-18=0$

**7.** $r^2-8r-33=0$ **8.** $t^2+13t+40=0$ **9.** $3u^2+7u-2=0$

**10.** $6z^2-9z+1=0$ **11.** $2a^2-6a+3=0$ **12.** $2x^2+3x+9=0$

**13.** $3t(t-2)=2$ **14.** $v(v+5)-10=0$ **15.** $3w(w-2)-8=0$

**16.** $6y^2-5y+2=0$ **17.** $2a^2+12a+5=0$ **18.** $16y^2+8y+1=0$

### Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

*In the following exercises, determine the number of solutions to each quadratic equation.*

**19.** $4x^2-5x+16=0$ **20.** $9v^2-15v+25=0$ **21.** $r^2+12r+36=0$

**22.** $6m^2+3m-5=0$ **23.** $4u^2-12u+9=0$ **24.** $3v^2-5v-1=0$

**25.** $4x^2-5x+16=0$ **26.** $5c^2+7c-10=0$ **27.** $8t^2-11t+5=0$

## 8.2 Applications of Quadratic Equations

We have solved number applications that involved consecutive even integers and consecutive odd integers by modeling the situation with linear equations. One set of even integers and one set of odd integers are shown below.

| **Consecutive even integers 64, 66, 68** | | **Consecutive odd integers 77, 79, 81** | |
|---|---|---|---|
| $n$ | $1^{st}$ even integer | $n$ | $1^{st}$ odd integer |
| $n+2$ | $2^{nd}$ consecutive even integer | $n+2$ | $2^{nd}$ consecutive odd integer |
| $n+4$ | $3^{rd}$ consecutive even integer | $n+4$ | $3^{rd}$ consecutive odd integer |

Some applications of consecutive odd integers or consecutive even integers are modeled by quadratic equations. The notation above will be helpful as you name the variables.

**Example 8.8:** The product of two consecutive odd integers is 195. Find the integers.

**Solution:**

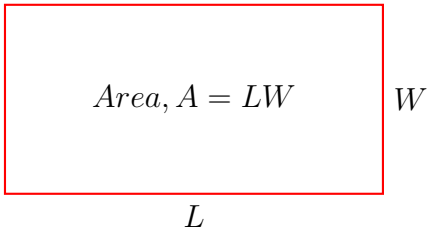
| | |
|---|---|
| **Step 1. Read** the problem. | |
| **Step 2. Identify** what we are looking for. | We are looking for two consecutive odd integers. |
| **Step 3. Name** what we are looking for. | Let $n =$ the first odd integer.<br>$n+2 =$ the next odd integer |
| **Step 4. Translate** into an equation. State the problem in one sentence. | "The product of two consecutive odd integers is 195." The product of the first odd integer and the second odd integer is 195. |
| Translate into an equation | $n(n+2) = 195$ |
| **Step 5. Solve** the equation. Distribute. | $n^2 + 2n = 195$ |
| Subtract 195 to get the equation in standard form. | $ax^2 + bx + c = 0$<br>$n^2 + 2n - 195 = 0$ |
| Identify the $a$, $b$, $c$ values. | $a = 1,\ b = 2, c = -195$ |
| Write the quadratic formula. | $n = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| Then substitute in the values of $a$, $b$, $c$. | $n = \dfrac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-195)}}{2 \cdot 1}$ |
| Simplify. | $n = \dfrac{-2 \pm \sqrt{4 + 780}}{2}$<br>$n = \dfrac{-2 \pm \sqrt{784}}{2}$ |
| Simplify the radical. | $n = \dfrac{-2 \pm 28}{2}$ |
| Rewrite to show two solutions. | $n = \dfrac{-2 + 28}{2}, n = \dfrac{-2 - 28}{2}$ |

| | |
|---|---|
| Solve each equation. | $n = \frac{26}{2} = 13$ and $n = \frac{-30}{2} = -15$ |
| There are two values of $n$ that are solutions. This will give us two pairs of consecutive odd integers for our solution. | **when** $n = 13$**:**<br>First odd integer: $n = 13$<br>next odd integer: $n + 2 = 13 = 2 = 15$<br><br>**when** $n = -15$**:**<br>First odd integer: $n = -15$<br>next odd integer: $n + 2 = -15 + 2 = -13$ |
| **Step 6. Answer** the question. | The two consecutive odd integers whose product is 195 are 13, 15, and 13, 15. |

**We will use the formula for the area of a rectangle to solve the next example.**

**Area of a Triangle:**

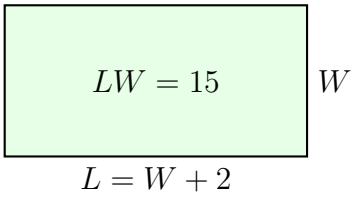
For a recatange with length $L$ and width $W$ , the area, $A$ , is given by the formula $A = LW$.




**Example 8.9:** A rectangular garden has an area 15 square feet. The length of the garden is two feet more than the width. Find the length and width of the garden.

**Solution:**

Let $L$ = Length of the garden
$W$ = Width of the garden
$A$ = Area of the garden
$A = LW$
Given: The length of the garden is two feet more than the width.
$L = W + 2$




Given that, Area, $A = 15$ square feet.

$A = 15 \Rightarrow LW = 15$

$\Rightarrow (W + 2)W = 15$

$\Rightarrow W^2 + 2W = 15$ Multiply

$\Rightarrow W^2 + 2W - 15 = 0$ Write the quadratic equation in the standard form

$\Rightarrow W = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Write the quadratic formula

$$\Rightarrow W = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-15)}}{2 \cdot 1} \quad \text{Substitute} a = 1, b = 2, c = -15$$

$$\Rightarrow W = \frac{-2 \pm \sqrt{64}}{2} \quad \text{Simplify}$$

$$\Rightarrow W = \frac{-2 \pm 8}{2} \quad \text{Simplify}$$

$$\Rightarrow W = \frac{-2 + 8}{2}, \ \frac{-2 - 8}{2} \quad \text{Find the two solutions}$$

$$\Rightarrow W = \frac{6}{2}, \ \frac{-10}{2} \quad \text{Simplify}$$

$$\Rightarrow W = 3, -5$$

Since $W$ is the width of the garden, it does not make sense for it to be negative. We eliminate that value for $W$.

Width of the garden, $W = 3$

Length of the garden, $L = W + 2 = 5$.

## 8.2 Section Exercises

### Solve Applications of the Quadratic Formula

*In the following exercises, solve by using the Quadratic Formula.*

**1.** The product of two consecutive integers is 72. Find the numbers.

**2.** The product of two consecutive odd numbers is 63. Find the numbers.

**3.** The product of two consecutive even numbers is 80. Find the numbers.

**4.** The product of two consecutive even numbers is 24. Find the numbers.

**5.** The product of two consecutive odd numbers is 35. Find the numbers.

**6.** The product of two consecutive odd numbers is 99. Find the numbers.

**7.** The product of two consecutive even numbers is 48. Find the numbers.

**8.** A triangle with area 45 square inches has a height that is two less than four times the width. Find the height and width of the triangle. [Hint: *For a triangle with base $b$ and height $h$ , the area, $A$ , is given by the formula $A = \frac{1}{2}bh$.*]

**9.** The width of a triangle is six more than twice the height. The area of the triangle is 88 square yards. Find the height and width of the triangle.

**10.** A rectangular sign board has area 30 square feet. The length of the sign is one foot more than the width. Find the length and width of the sign board.

**11.** A rectangular lawn has area 180 square feet. The width of the patio is three feet less than the length. Find the length and width of the lawn.

# Chapter 9

# COORDINATE GEOMETRY

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**Learning outcome covered:**

**f.** Use coordinate plane to solve algebraic and geometric problem, and understand geometric concepts such as equation of circle, perpendicular, parallel, and tangent lines.

**g.** Use the three types of symmetry of an equation to sketch its graph.

**Learning Objectives**

**By the end of this chapter, the students will be able to:**

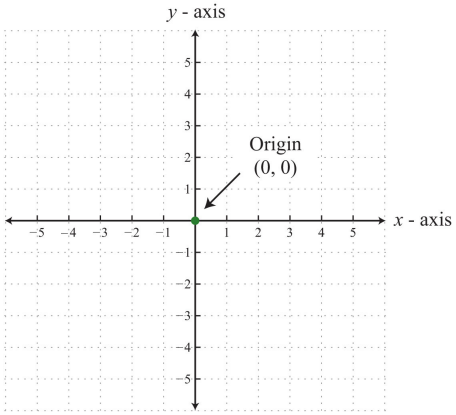
- Plot points using the rectangular coordinate system.
- Calculate the distance between any two points in the rectangular coordinate plane.

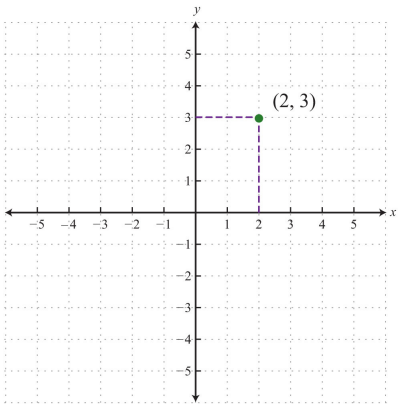
- Identify and find the slope of a line.
- Find the equation of the line using the slope and y-intercept.
- Find the equation of the line using point-slope form.
- Determine the slopes of parallel and perpendicular lines.
- Write the equation of a circle in standard form
- Test Equations for Symmetry

## 9.1 Rectangular Coordinate System

### 9.1.1 Rectangular Coordinate System

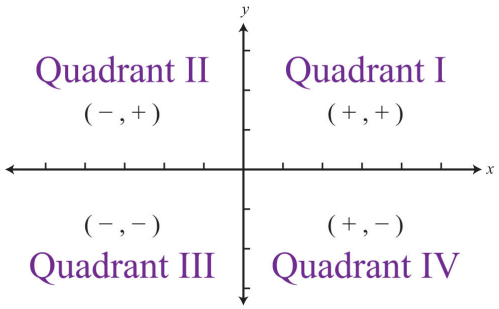
The rectangular coordinate system consists of two real number lines that intersect at a right angle. The horizontal number line is called the $x$-axis, and the vertical number line is called the $y$-axis. These two number lines define a flat surface called a plane, and each point on this plane is associated with an ordered pair of real numbers $(x, y)$. The first number is called the **$x$-coordinate**, and the second number is called the **$y$-coordinate**. The intersection of the two axes is known as the **origin**, which corresponds to the point (0, 0).



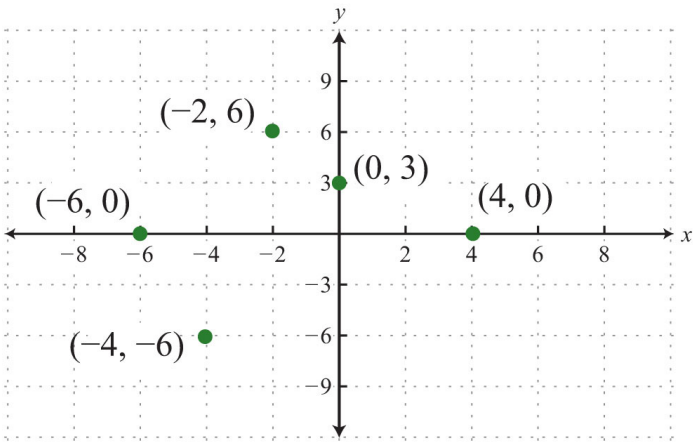

An ordered pair $(x, y)$ represents the position of a point relative to the origin. The $x$-coordinate represents a position to the right of the origin if it is positive and to the left of the origin if it is negative. The $y$-coordinate represents a position above the origin if it is positive and below the origin if it is negative. Using this system, every position (point) in the plane is uniquely identified. This system is often called the Cartesian coordinate system.

The $x$- and $y$-axes break the plane into four regions called quadrants, named using roman numerals I, II, III, and IV, as pictured. In quadrant I, both coordinates are positive. In quadrant II, the $x$-coordinate is negative and the $y$-coordinate is positive. In quadrant III, both coordinates are negative. In quadrant IV, the $x$-coordinate is positive and the $y$-coordinate is negative.




**Example 9.1:** Plot the set of ordered pairs: {(4, 0), (6, 0), (0, 3), (2, 6), (4, 6)}

**Solution:** Each marked points on the $x$-axis represents 2 units and each marked points on the $y$-axis represents 3 units.



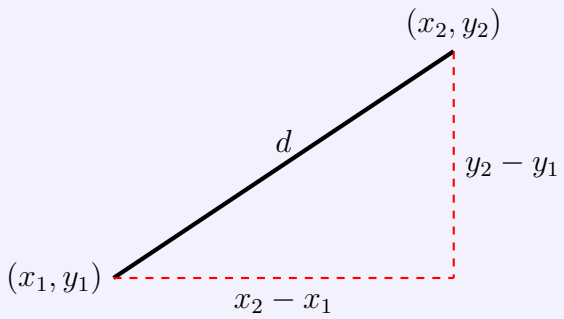

### 9.1.2 The distance between two points

The distance between two points is the length of the straight line between those two points. The formula to find the distance between two points is given below.

**Distance formula**

Given two points, $(x_1, y_1)$ and $(x_2, y_2)$, then the distance, $d$, between them is given by the distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



**Example 9.2:** Calculate the distance between (3, 1) and (2, 4).

**Solution:** Use the distance formula.

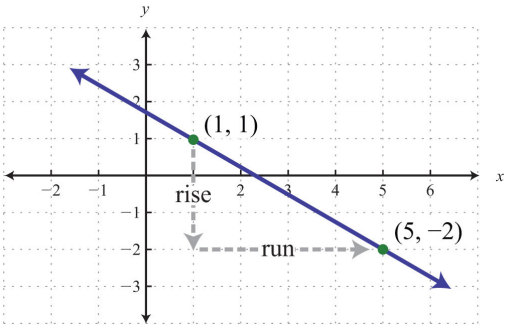
$$\begin{array}{lll} & (x_1, y_1) \quad (x_2, y_2) & \\ & (-3, -1) \quad (-2, 4) & \\ d = & \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} & \text{Write the distance formula.} \\ = & \sqrt{(-2 - (-3))^2 + (4 - (-1))^2} & \text{Substitute.} \\ = & \sqrt{(-2 + 3)^2 + (4 + 1)^2} & \text{Simplify.} \\ = & \sqrt{1^2 + 5^2} & \\ = & \sqrt{1 + 25} & \\ = & \sqrt{26} & \end{array}$$

### 9.1.3 Slope of a Straight line

The steepness of any incline can be measured as the ratio of the vertical change to the horizontal change. In mathematics, we call the incline of a line the slope and use the letter m to denote it. The vertical change is called the rise and the horizontal change is called the run.

$$\text{Slope, } m = \frac{\text{Vertical Change}}{\text{Horizontal Change}} = \frac{\text{Rise}}{\text{Run}}$$

**Example 9.3:** Find the slope of the given line:



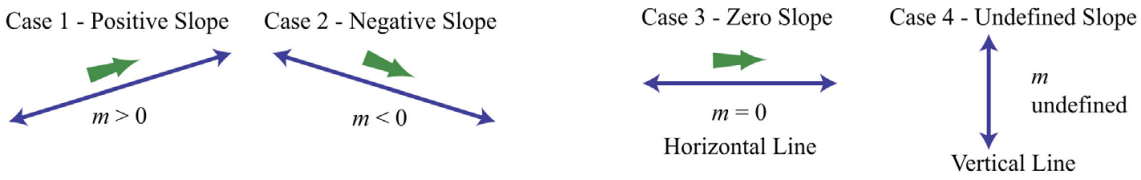

**Solution:** From the given points on the graph, count 3 units down and 4 units right.

$$\text{Slope, } m = \frac{\text{Rise}}{\text{Run}} = \frac{\text{-3 units}}{\text{4 units}} = -\frac{3}{4}$$

Here we have a negative slope, which means that for every 4 units of movement to the right, the vertical change is 3 units downward.

There are four geometric cases for the value of the slope.Reading the graph from left to right, we see that lines with an upward incline have positive slopes and lines with a downward

incline have negative slopes.




**Horizontal & Vertical line**

❋ For a horizontal line, Rise $= 0$. Hence, slope, $m = \dfrac{Rise}{Run} = 0 = \dfrac{0}{Run} = 0$.

❋ For a vertical line, Run $= 0$. Hence, slope, $m = \dfrac{Rise}{Run} = \dfrac{Rise}{0}$, **undefined**.

### Slope of the line joining two points

Given any two points $(x_1, y_1)$ and $(x_2, y_2)$, we can obtain the rise and run by subtracting the corresponding coordinates. This leads us to the slope formula.

**Slope of the line joining two points**

Given any two points $(x_1, y_1)$ and $(x_2, y_2)$, the slope is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

**Example 9.4:** Find the slope of the line passing through (3, 5) and (2, 1).

**Solution:** Given (3, 5) and (2, 1), calculate the difference of the $y$-values divided by the difference of the $x$-values. Since subtraction is not commutative, take care to be consistent when subtracting the coordinates.

$$\begin{array}{rl} & (x_1, y_1) \quad (x_2, y_2) \\ & (-3, -5) \quad (2, 1) \\ m = & \dfrac{y_2 - y_1}{x_2 - x_1} \\ = & \dfrac{1 - (-5)}{2 - (-3)} = \dfrac{1 + 5}{2 + 3} = \dfrac{6}{5} \end{array}$$

## 9.1 Section Exercises

### Ordered pairs

*Plot the given set of ordered pairs.*

**1.** $\{(-4,5),(-1,1),(-3,-2),(5,-1)\}$

**2.** $\{(-2,5),(10,0),(2,-5),(6,-10)\}$

**3.** $\{(-8,3),(-4,6),(0,-6),(6,9)\}$

**4.** $\{(-3.5,0),(-1.5,2),(0,1.5),(2.5,-1.5)\}$

*State the quadrant in which the given point lies.*

**5.** $(-3,2)$

**6.** $(5,7)$

**7.** $(-12,-15)$

**8.** $(7,-8)$

**9.** $(-3.8,4.6)$

**10.** $(-18,-58)$

### Distance Formula

*Calculate the distance between the given two points.*

**11.** $(-5,3)$ and $(-9,6)$

**12.** $(6,-2)$ and $(-2,4)$

**13.** $(0,0)$ and $(5,12)$

**14.** $(-6,-8)$ and $(0,0)$

**15.** $(-7,8)$ and $(5,-1)$

**16.** $(1,2)$ and $(4,3)$

**17.** $(2,-4)$ and $(-3,-2)$

**18.** $(-7,-3)$ and $(2,6)$

**19.** $(0,1)$ and $(1,0)$

### Slope Formula

*Find the slope of the line joining the given two points.*

**20.** $(3,2)$ and $(5,1)$

**21.** $(7,8)$ and $(-3,5)$

**22.** $(2,-3)$ and $(-3,2)$

**23.** $(-3,2)$ and $(7,-5)$

**24.** $(-1,-6)$ and $(3,2)$

**25.** $(5,3)$ and $(4,12)$

**26.** $(-3,1)$ and $(-14,1)$

**27.** $(-4,-4)$ and $(5,5)$

**28.** $(-2,3)$ and $(-2,-4)$

**29.** Find $b$ if the slope of the line passing through $(-2,3)$ and $(4,b)$ is 12.

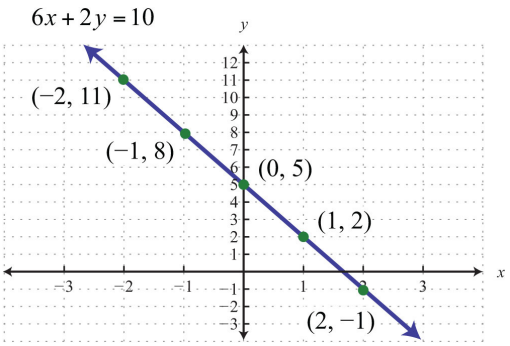
**30.** Find $b$ if the slope of the line passing through $(5,b)$ and $(-4,2)$ is 0.

**31.** Find $a$ if the slope of the line passing through $(-3,2)$ and $(a,5)$ is undefined.

## 9.2 Straight Lines

A linear equation with two variables has standard form $ax + by = c$, where $a$, $b$, and $c$ are real numbers and $a$ and $b$ are not both 0. Since the solutions to linear equations are ordered pairs, they can be graphed using the rectangular coordinate system. The set of all solutions to a linear equation can be represented on a rectangular coordinate plane using a **straight line** connecting at least two points.

To illustrate this, plot five ordered pair solutions, (2, 11), (1, 8), (0, 5), (1, 2), (2, 1) to the linear equation $6x + 2y = 10$. Draw a line through the points with a straightedge, and add arrows on either end to indicate that the graph extends indefinitely. The resulting line represents all solutions to $6x + 2y = 10$.

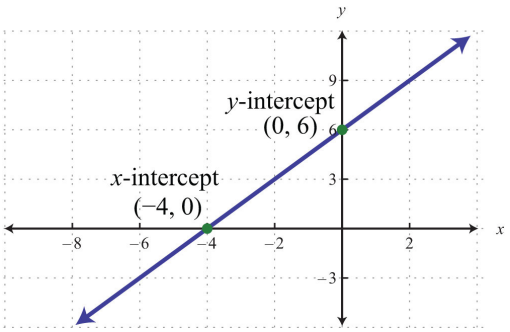



*Hence, any linear equation $ax + by = c$ with two variables where $a$, $b$, and $c$ are real numbers and $a$ and $b$ are not both 0 always represents a straight line.*

### 9.2.1 Slope - Intercept Form of a Line

**Definition of $x$ - and $y$ - Intercepts**

The $x$ - intercept is the point where the graph of a line intersects the $x$ - axis. The $y$ - intercept is the point where the graph of a line intersects the $y$ - axis. These points have the form $(x, 0)$ and $(0, y)$, respectively.




**To find the $x$ - and $y$ - intercepts:**

- To find the $x$ - intercept, set $y = 0$ and determine the corresponding $x$ - value.
- To find the $y$ - intercept, set $x = 0$ and determine the corresponding $y$ - value.

**Slope - Intercept Form of a Line**

The equation of any non-vertical line can be written in slope - intercept form $y = mx + b$. In this form, we can identify the slope, $m$, and the $y-$intercept, $(0, b)$.

**Example 9.5:** Determine the slope and $y-$intercept: $y = -\frac{4}{5}x + 7$

**Solution:** In this form, the coefficient of $x$ is the slope, and the constant is the $y$ - coordinate of the $y$ - intercept. Therefore, by inspection, we have

$$y = -\frac{4}{5}x + 7$$

$y-$ intercept, (0, 7)

Slope, $m = -\frac{4}{5}$

The $y$ - intercept is (0, 7), and the slope is $m = -\frac{4}{5}$.

---

**Example 9.6:** Find the equation of a line with slope $m = -\frac{5}{8}$ and $y$ - intercept (0, 1).

**Solution:** The given $y$ - intercept implies that $b = 1$. Substitute the slope $m$ and the $y$ - value of the $y$ - intercept $b$ into the equation $y = mx + b$.

$$y = mx + b$$

$$y = -\frac{5}{8}x + 1$$

Answer: $y = -\frac{5}{8}x + 1$

---

### 9.2.2 Point - Slope form of a line

**Finding equation of line Using a point and the slope**

Given any point on a line and its slope, we can find the equation of that line. Begin by applying the slope formula with a given point $(x_1, y_1)$ and a variable point $(x, y)$.

**Point - Slope form of a line**

The equation of any non-vertical line with slope $m$ and passing through the point $(x_1, y_1)$ can be written in the point - slope form $y-y_1 = m(x-x_1)$.

**Example 9.7:** Find the equation of the line with slope $m = \frac{1}{2}$ passing through (4, 1).

**Solution:** Use point - slope form, where $m = \frac{1}{2}$ and $(x_1, y_1) = (4, -1)$.

$$\begin{aligned} y - y_1 &= m(x-x_1) \\ y - (-1) &= \frac{1}{2}(x - 4) \\ y + 1 &= \frac{1}{2}x - 2 \\ y &= \frac{1}{2}x - 3 \end{aligned}$$

---

### 9.2.3 Standard Equation of Straight Lines

**Standard equation of a straight line**

The standard equation of a straight line is $ax + by + c = 0$, where $a$, $b$ and $c$ are real numbers and $a$ and $b$ are not both 0.

For the straight line $ax + by + c = 0$,

- Slope, $m = -\dfrac{a}{b}$
- $x$- intercept $= \left(-\dfrac{c}{a}, 0\right)$
- $y$- intercept $= \left(0, -\dfrac{c}{b}\right)$

**Example 9.8:** Find the slope, $x$-inercept and $y$- intercept of the line $2x - 3y + 4 = 0$.

**Solution:** $a = 2$, $b = -3$ and $c = 4$.

$$\text{Slope, } m = -\frac{a}{b} = -\frac{2}{-3} = \frac{2}{3}$$

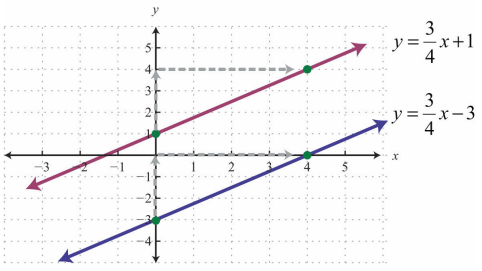
$$x - \text{intercept} = \left(-\frac{c}{a}, 0\right) = \left(-\frac{4}{2}, 0\right) = (-2, 0)$$

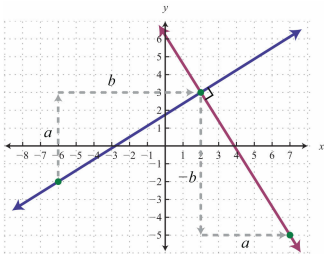
$$y - \text{intercept} = \left(0, -\frac{c}{b}\right) = \left(0, -\frac{4}{-3}\right) = \left(0, \frac{4}{3}\right)$$

### 9.2.4 Parallel lines and Perpendicular Lines

***Parallel lines*** are lines in the same plane that never intersect. Two lines in the same plane, with slopes $m_1$ and $m_2$, are parallel if their slopes are the same, $m_1 = m_2$.

***Perpendicular lines*** are lines in the same plane that intersect at right angles (90 degrees). Two lines in the same plane, with slopes $m_1$ and $m_2$, are perpendicular if the product of their slopes is $-1$: $m_1 \cdot m_2 = -1$. That is, $m_2 = -\dfrac{1}{m_1}$.




- **If $m$ is the slope of a straight line, then**
  - the slope of a parallel line is, $m_{\parallel} = m$.
  - the slope of a perpendicular line is, $m_{\perp} = -\dfrac{1}{m}$.

⚽ **If $m_1$ and $m_2$ are the slopes of two straight lines $L_1$ and $L_2$, then**

- ✤ $L_1$ and $L_2$ are parallel if $m_1 = m_2$.
- ✤ $L_1$ and $L_2$ are perpendicular $m_1 \cdot m_2 = -1$.

**Example 9.9:** Determine the slope of a line parallel to $y = -5x + 3$.

**Solution:** Since the given line is in slope - intercept form, we can see that its slope is m=5. Thus the slope of any line parallel to the given line must be the same, $m_{\parallel} = -5$.

---

**Example 9.10:** Determine the slopes of the lines that are parallel and perpendicular to $3x-7y = 21$.

**Solution:** Write the equation in the standard form.

The equation of the straight line is $3x-7y - 21 = 0$. $a = 3,$ $b = -7,$ $c = -21$

Slope, $m = -\dfrac{a}{b} = -\dfrac{3}{-7} = \dfrac{3}{7}$

Slope of the parallel line is, $m_{\parallel} = m = \dfrac{3}{7}$

Slope of the perpendicular line is, $m_{\perp} = -\dfrac{1}{m} = -\dfrac{7}{3}$

---

**Example 9.11:** Determine if the lines $2x - 3y + 1 = 0$ and $9x + 6y - 5 = 0$ are parallel, perpendicular, or neither.

**Solution:**

Slope of line $L_1 : 2x - 3y + 1 = 0$ is $a = 2,\ b = -3,\ c = 1$

$$m_1 = -\frac{a}{b} = -\frac{2}{-3} = \frac{2}{3}$$

Slope of line $L_2 : 9x + 6y - 5 = 0$ is $a = 9,\ b = 6,\ c = -5$

$$m_2 = -\frac{a}{b} = -\frac{9}{6} = -\frac{3}{2}$$

$m_1 \neq m_2$. The lines are not parallel.

$m_1 \times m_2 = \dfrac{2}{3} \cdot -\dfrac{3}{2} = -1$ The lines are perpendicular.

---

## 9.2 Section Exercises

*Find the $x-$ intercepts and $y$ - intercepts.*

**1.** $5x-4y=20$ **2.** $-2x+7y=-28$ **3.** $x-y=3$

**4.** $y=6$ **5.** $x=-1$ **6.** $y=mx+b$

**Slope - Intercept Form**

*Express the given linear equation in slope - intercept form and identify the slope and $y$ - intercept.*

**7.** $6x-5y=30$ **8.** $-2x+7y=28$ **9.** $x-3y=18$

**10.** $2x-3y=0$ **11.** $-6x+3y=0$ **12.** $9x-y=17$

*Given the slope and $y$ - intercept, determine the equation of the line.*

**13.** $m=4;(0,-1)$ **14.** $m=-3;(0,9)$ **15.** $m=0;(0,-1)$

**16.** $m=\frac{1}{2};(0,5)$ **17.** $m=-\frac{2}{3};(0,-4)$ **18.** $m=5;(0,0)$

*Use the point - slope formula to find the equation of the line passing through the two points.*

**19.** $(-4,0),(0,5)$ **20.** $(-1,2),(0,3)$ **21.** $(-3,-2),(3,2)$

**22.** $(-3,-1),(3,3)$ **23.** $(1,5),(0,5)$ **24.** $(-8,0),(6,0)$

**Parallel and Perpendicular Lines**

*Determine the slopes of the lines that are parallel and perpendicular to the given lines:*

**25.** $y=-34x+8$ **26.** $-2x+7y=28$ **27.** $y=4x+4$

**28.** $4x+3y=0$ **29.** $-2x+7y=14$ **30.** $-x-y=15$

**31.** $y=5$ **32.** $x=-12$ **33.** $x-y=0$

*Determine if the lines are parallel, perpendicular, or neither.*

**34.** $\begin{cases} y=\frac{2}{3}x+3 \\ y=\frac{2}{3}x-3 \end{cases}$ **35.** $\begin{cases} y=\frac{3}{4}x-1 \\ y=\frac{4}{3}x+3 \end{cases}$ **36.** $\begin{cases} y=-2x+1 \\ y=\frac{1}{2}x+8 \end{cases}$

**37.** $\begin{cases} x-y=7 \\ 3x+3y=2 \end{cases}$ **38.** $\begin{cases} 2x-6y=4 \\ -x+3y=-2 \end{cases}$ **39.** $\begin{cases} 3x-5y=15 \\ 5x+3y=9 \end{cases}$

## 9.3 Circle

**Circle**

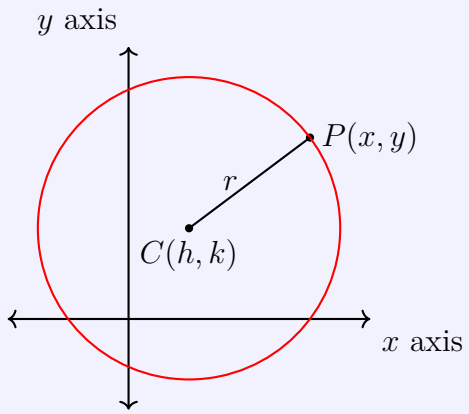
A circle is all points in a plane at a fixed distance from a given point in the plane. The given point is called the center, $(h, k)$, and the fixed distance is called the radius, $r$, of the circle.

We look at a circle in the rectangular coordinate system. The radius is the distance from the center, $(h, k)$, to a point on the circle, $(x, y)$.

### 9.3.1 Standard Form of the Equation a Circle

**Standard Form of the Equation of a Circle**

The standard form of the equation of a circle with center $(h, k)$, and radius $r$, is $(x - h)^2 + (y - k)^2 = r^2$.




**Example 9.12:** Write the standard form of the equation of the circle with radius 3 and center (0, 0).

**Solution:** The standard form of the equation of a circle is $(x - h)^2 + (y - k)^2 = r^2$.
Substitute the values $r = 3$, $h = 0$, and $k = 0$.

$$\text{The equation of the circle is } (x - 0)^2 + (y - 0)^2 = 3^2 \Rightarrow \qquad x^2 + y^2 = 9$$

In the last example, the center was (0, 0). Notice what happened to the equation. Whenever the center is (0, 0), the standard form becomes $x^2 + y^2 = r^2$.

**Example 9.13:** Write the standard form of the equation of the circle with radius 2 and center (1, 3).

**Solution:** The standard form of the equation of a circle is $(x - h)^2 + (y - k)^2 = r^2$.
Substitute the values $r = 2$, $h = -1$, and $k = 3$.

$$\text{The equation of the circle is } (x - (-1))^2 + (y - 3)^2 = 2^2$$
$$(x + 1)^2 + (y - 3)^2 = 4$$

In the next example, the radius is not given. To calculate the radius, we use the Distance Formula with the two given points.

**Example 9.14:** Write the standard form of the equation of the circle with center (2, 4) and passing through the point (2, 1).

**Solution:** The radius is the distance from the center to any point on the circle so we can use the distance formula to calculate it. We will use the center $(2, 4)$ and point $(-2, 1)$.

Use the Distance Formula to find the radius.

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{Distance Formula}$$

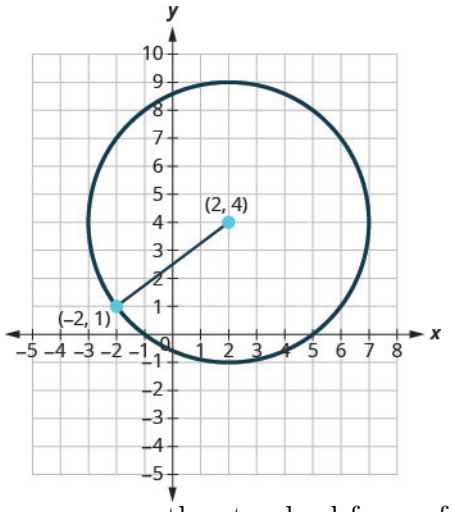
$$= \sqrt{(-2 - 2)^2 + (1 - 4)^2} \qquad \begin{matrix}(x_1,y_1) & (x_2,y_2)\\ (2,4) & (-2,1)\end{matrix}$$

$$= \sqrt{(-4)^2 + (-3)^2} \quad \text{Simplify}$$

$$= \sqrt{16 + 9}$$

$$= \sqrt{25} = 5$$

$$= 5$$




Now that we know the radius, r = 5, and the center, (2, 4), we can use the standard form of the equation of a circle to find the equation.

Use the standard form of the equation of a circle. $(x - h)^2 + (y - k)^2 = r^2$

The equation of the circle is, $(x - 2)^2 + (y - 4)^2 = 5^2$

$$(x - 2)^2 + (y - 4)^2 = 25$$

---

**Example 9.15:** Find the center and radius of the $(x + 2)^2 + (y - 1)^2 = 9$.

**Solution:**

$$(x + 2)^2 \ + \ (y - 1)^2 = 9$$

Use the standard form of the equation of a circle. $(x - h)^2 \ + \ (y - k)^2 = r^2$

Identify the center, $(h, k)$ and radius, $r$.

$$(x - (-2))^2 + (y - 1)^2 = 3^2$$

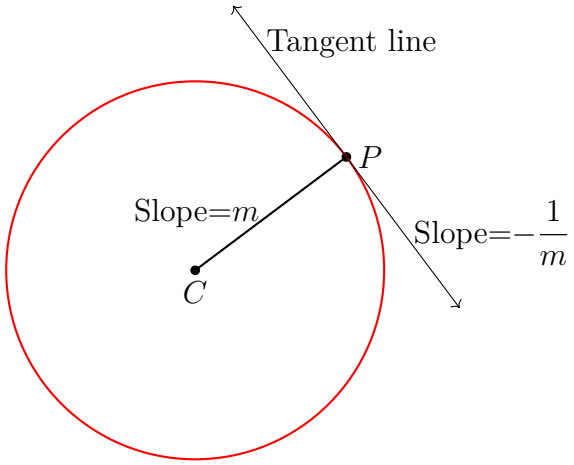
Center: $(-2, 1)$ Radius: 3

---

### 9.3.2 Tangent lines

**Tangent:** A tangent to a circle is a line which intersects the circle in exactly one point. A tangent to a circle can also be defined as a straight line which touches the circle at only one point. This point is called the point of tangency or the point of contact.

The tangent to a circle is always perpendicular to the radius at the point of contact.




**Finding the equation of the tangent line at a point on the circle.**

- Find the center of the circle, $C = (h, k)$.
- Take the point as $P = (x_1, y_1)$.
- Find the slope of the radius $PC$, $m = \dfrac{y_1 - k}{x_1 - h}$.
- The tangent line at $P$ is perpendicular to the radius $PC$. Therefore, slope of the tangent is $m' = -\dfrac{1}{m}$.
- The tangent at $P$ is the line passing through $(x_1, y_1)$ and having slope $m' = -\dfrac{1}{m}$.
- Equation of the tangent is $y - y_1 = m'(x - x_1)$.

**Example 9.16:** Find the equation of the tangent to the circle $(x-2)^2 + (y+1)^2 = 25$ at the point $(5, 3)$.

**Solution:**

Let $C$ be the center and $P$ be the point. $C$: $(h, k) = (2, -1)$, $P$: $(x_1, y_1) = (5, 3)$

Slope of the radius $CP$ is, $m = \dfrac{3-(-1)}{5-2} = \dfrac{4}{3}$ $m = \dfrac{y_1 - k}{x_1 - h}$

Slope of the tangent line is, $m' = -\dfrac{1}{m} = -\dfrac{3}{4}$

Equation of the tangent at $(5, 3)$ is

$$\begin{aligned} y - y_1 &= m'(x - x_1) \\ y - 3 &= -\frac{3}{4}(x - 5) \\ 4(y - 3) &= -3(x - 5) \\ 4y - 12 &= -3x + 15 \\ 3x + 4y - 27 &= 0 \end{aligned}$$

## 9.3 Section Exercises

### Write the Equation of a Circle in Standard Form

*In the following exercises, write the standard form of the equation of the circle with the given radius and center (0, 0).*

**1.** Radius: 7

**2.** Radius: 9

**3.** Radius: $\sqrt{2}$

**4.** Radius: $\sqrt{5}$

*In the following exercises, write the standard form of the equation of the circle with the given radius and center.*

**5.** Radius: 1, center: $(3, 5)$

**6.** Radius: 10, center: $(-2, 6)$

**7.** Radius: $\sqrt{15}$, center: $(0, 0)$

**8.** Radius: 5, center: $(-1, 0)$

**9.** Radius: 2.5, center: $(1.5, -3.5)$

**10.** Radius: $3\sqrt{2}$, center: $(-5.5, -6.5)$

*For the following exercises, write the standard form of the equation of the circle with the given center with point on the circle.*

**11.** Center $(3, -2)$ with point $(3, 6)$

**12.** Center $(6, -6)$ with point $(2, -3)$

**13.** Center $(0, 0)$ with point $(-6, 8)$

**14.** Center $(-3, 4)$ with point $(0, 0)$

**15.** Center $(4, 4)$ with point $(2, 2)$

**16.** Center $(-5, 6)$ with point $(-2, 3)$

### Find the center and radius

*In the following exercises, find the center and radius of each circle.*

**17.** $(x + 5)^2 + (y + 3)^2 = 1$

**18.** $(x - 2)^2 + (y - 3)^2 = 9$

**19.** $x^2 + (y + 2)^2 = 75$

**20.** $(x + 2)^2 + (y + 5)^2 = 4$

**21.** $(x - 4)^2 + (y + 2)^2 = 32$

**22.** $(x - 1)^2 + y^2 = 36$

### Tangent lines

**23.** Find the equation of the tangent to the circle $(x+9)^2+(y+2)^2=125$ at $(1,3)$.

**24.** Find the equation of the tangent to the circle $x^2+y^2=29$ at $(-2,5)$.

**25.** Find the equation of the tangent to the circle $(x-3)^2+y^2-8=0$ at $(1,-2)$.

**26.** Find the equation of the tangent to the circle $(x+5)^2+(y-7)^2=20$ at $(-7,3)$.

**27.** Find the slope of the tangent to the circle $(x-1)^2+(y+1)^2=5$ at $(2,-3)$.

**28.** Find the slope of the tangent to the circle $(x-2)^2+(y-3)^2=18$ at $(-1,6)$.

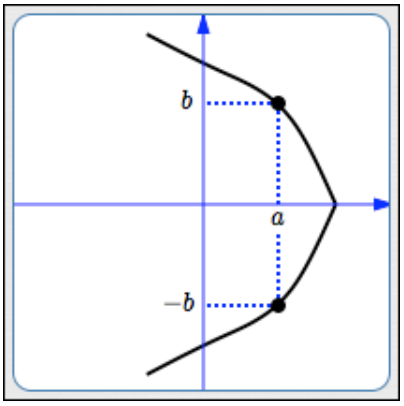
## 9.4 Testing Equations for Symmetry

A knowledge of symmetry can increase your efficiency when working with graphs. This section discusses:

- ✠ symmetry about the $x$-axis
- ✠ symmetry about the $y$-axis
- ✠ origin symmetry

**Definition: $x$-axis symmetry**

A graph has symmetry about the $x$-axis if and only if whenever $(a, b)$ is on the graph, so is $(a, -b)$.

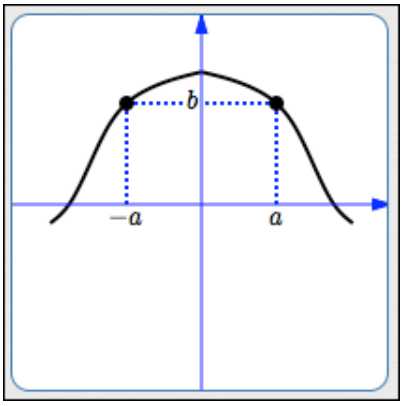



**Test for $x$-axis symmetry**

- Replace every $y$ by $-y$ in the equation.
  - ✓ **symmetric with respect to the $x$- axis** if the equation is the same.
  - ✗ **not symmetric with respect to the $x$- axis** if the equation is not the same.

**Definition: $y$-axis symmetry**

A graph has symmetry about the $y$-axis if and only if whenever $(a, b)$ is on the graph, so is $(-a, b)$.

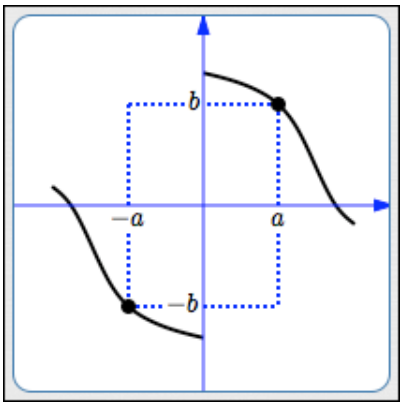



**Test for $y$-axis symmetry**

- Replace every $x$ by $-x$ in the equation.
  - ✓ **symmetric with respect to the $y$- axis** if the equation is the same.
  - ✗ **not symmetric with respect to the $y$- axis** if the equation is not the same.

**Definition: Origin symmetry**

A graph has symmetry about the origin if and only if whenever $(a, b)$ is on the graph, so is $(-a, -b)$.




**Test for Origin symmetry**

- Replace every $x$ by $-x$ and $y$ by $-y$ in the equation.
  - ✓ **symmetric with respect to the origin** if the equation is the same.
  - ✗ **not symmetric with respect to the origin** if the equation is not the same.

**Example 9.17:** Test $2x - 3y^2 = 10$ for various symmetries.
**Solution:** The equation is $2x - 3y^2 = 10 \quad \cdots (\bigstar)$

| | |
|---|---|
| $x$ - Axis | ➥ Replace $y$ with $-y$.<br>$2x - 3(-y)^2 = 10$<br>$2x - 3y^2 = 10$<br>✣ same equation $(\bigstar)$<br>✣ The graph is symmetric with respect to the $x$ - axis. |
| $y$ - Axis | ➥ Replace $x$ with $-x$.<br>$2(-x) - 3y^2 = 10$<br>$-2x - 3y^2 = 10$<br>$2x + 3y^2 = -10$<br>✣ **not** the same equation $(\bigstar)$<br>✣ The graph is **not symmetric** with respect to the $y$ - axis. |
| Origin | ➥ Replace $x$ with $-x$ and $y$ with $-y$.<br>$2(-x) - 3(-y)^2 = 10$<br>$-2x - 3y^2 = 10$<br>$2x + 3y^2 = -10$<br>✣ **not** the same equation $(\bigstar)$<br>✣ The graph is **not symmetric** with respect to the origin. |

**Example 9.18:** Test $5x^2 - 3y^2 = 15$ for various symmetries.
**Solution:** The equation is $5x^2 - 3y^2 = 15 \quad \cdots (\bigstar)$

| | |
|---|---|
| $x$ - Axis | ➥ Replace $y$ with $-y$.<br>$5x^2 - 3(-y)^2 = 15$<br>$5x^2 - 3y^2 = 15$<br>✜ the same equation (★)<br>✜ The graph is symmetric with respect to the $x$ - axis. |
| $y$ - Axis | ➥ Replace $x$ with $-x$.<br>$5(-x)^2 - 3y^2 = 15$<br>$5x^2 - 3y^2 = 15$<br>✜ the same equation (★)<br>✜ The graph is symmetric with respect to the $y$ - axis. |
| Origin | ➥ Replace $x$ with $-x$ and $y$ with $-y$.<br>$5(-x)^2 - 3(-y)^2 = 15$<br>$5x^2 - 3y^2 = 15$<br>✜ the same equation (★)<br>✜ The graph is symmetric with respect to the origin. |

## 9.4 Section Exercises

**1.** Suppose that the point $(-2, 4)$ lies on a graph that has $x$-axis symmetry. What other point must lie on the graph?

**2.** Suppose that the point $(0, 7)$ lies on a graph that has $x$-axis symmetry. What other point must lie on the graph?

**3.** Suppose that the point $(-2, 9)$ lies on a graph that has $y$-axis symmetry. What other point must lie on the graph?

**4.** Suppose that the point $(4, -3)$ lies on a graph that has $y$-axis symmetry. What other point must lie on the graph?

**5.** Suppose that the point $(1, 10)$ lies on a graph that has origin symmetry. What other point must lie on the graph?

**6.** Suppose that the point $(-9, -1)$ lies on a graph that has origin symmetry. What other point must lie on the graph?

**7.** Suppose that the point $(6, -5)$ lies on a graph that has origin symmetry. What other point must lie on the graph?

*Test the following graphs for various symmetries.*

**8.** $y = 3x + 5$

**9.** $2y = x^2 - 6$

**10.** $x + y^2 = 4$

**11.** $3x^2 - 5xy + 2y^2 = 25$

**12.** $x^3 - 2xy - xy^2 = 0$

**13.** $-4x^4y = 8x^6y^7$

**14.** $x^2 + y^2 = 16$

**15.** $y = x^3$

# Module II

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# Chapter 10

# Functions and their Graphs

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10.4.6 Finding the $x-$and $y-$Intercepts of a Quadratic Function

10.4.7 Sketch the Graph of the Quadratic Function


**Learning outcomes covered:**

(p) `Demonstrate understanding of the definition of a function and its graph.`

(q) `Solve quadratic equations using quadratic formula.`

### Learning Objectives

**By the end of this chapter, the students will be able to:**

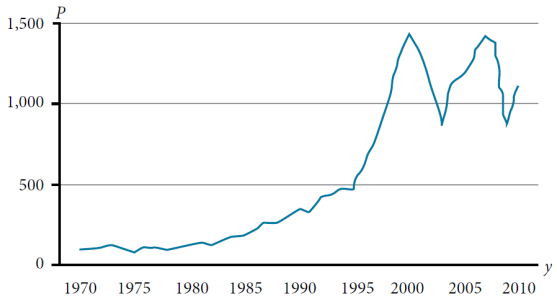
- Determine whether a relation represents a function.
- Find the domain and range of the function.
- Find the value of a function.
- Determine whether a function is one-to-one.

- Use the vertical line test to identify functions
- Use horizontal line test to identify one-to-one function.
- Find the inverse function and composition of functions.
- Solve quadratic equations using the quadratic Formula.
- Sketch the graph of linear and quadratic functions.

## Introduction

Toward the end of the twentieth century, the values of stocks of Internet and technology companies rose dramatically. As a result, the Standard and Poor's stock market average rose as well.

This figure tracks the value of that initial investment of just under $100 over the 40 years. It shows that an investment that was worth less than $500 until about 1995 skyrocketed up to about $1,100 by the beginning of 2000 . That five-year period became known as the "dot-com bubble" because so many Internet startups were formed. As bubbles tend to do, though, the dot-com bubble eventually burst. Many companies grew too fast and then suddenly went out of business. The result caused the sharp decline represented on the graph beginning at the end of 2000 .




Standard and Poor's Index with dividends reinvested
credit "graph": modification of work by MeasuringWorth

Notice, as we consider this example, that there is a definite relationship between the year and stock market average. For any year we choose, we can determine the corresponding value of the stock market average. In this chapter, we will explore these kinds of relationships and their properties.

## 10.1 Functions

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

A relation is a set of ordered pairs. The set of the first components of each ordered pair is called the domain of the relation and the set of the second components of each ordered pair is called the range of the relation. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice the first.

$$\{(1,2),(2,4),(3,6),(4,8),(5,10)\}$$

The domain is $\{1,2,3,4,5\}$ and the range is $\{2,4,6,8,10\}$.

Note that the values in the domain are also known as an **input values**, or values of the **independent variable**, and are often labeled with the lowercase letter $x$. Values in the range are also known as an **output values**, or values of the **dependent variable**, and are often labeled with the lowercase letter $y$.

A function $f$ is a relation that assigns a single value in the range to each value in the domain. In other words, no $x$-values are used more than once.

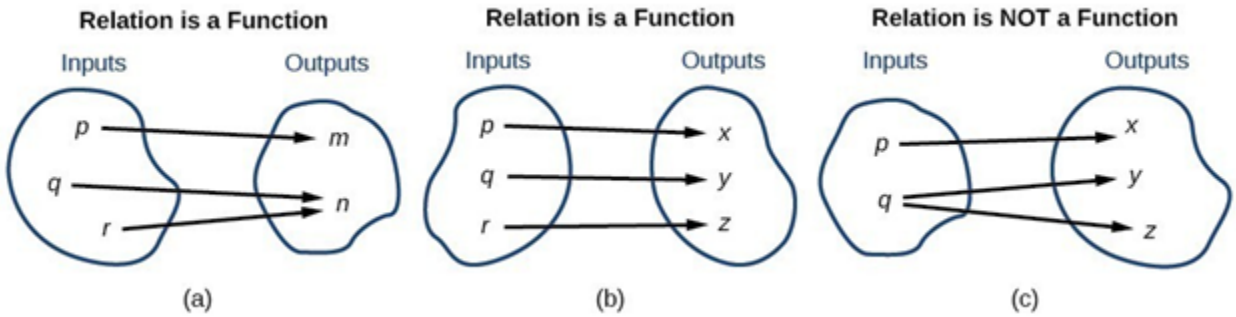
For our example that relates the first five natural numbers to numbers double their values, this relation is a function because each element in the domain, $\{1, 2, 3, 4, 5\}$ is paired with exactly one element in the range, $\{2, 4, 6, 8, 10\}$.

Now let's consider the set of ordered pairs that relates the terms "even" and "odd" to the first five natural numbers. It would appear as

$$\{(odd, 1), (even, 2), (odd, 3), (even, 4), (odd, 5)\}.$$

Notice that each element in the domain, $\{even, odd\}$ is not paired with exactly one element in the range, $\{1, 2, 3, 4, 5\}$. For example, the term "odd" corresponds to three values from the domain, $\{1, 3, 5\}$ and the term "even" corresponds to two values from the range,$\{2, 4\}$. This violates the definition of a function, so this relation is not a function.

This image compares relations that are functions and not functions.




(a) This relationship is a function because each input is associated with a single output. Note that input q and r both give output n.

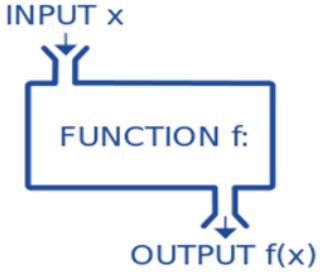
(b) This relationship is also a function. In this case, each input is associated with a single output.

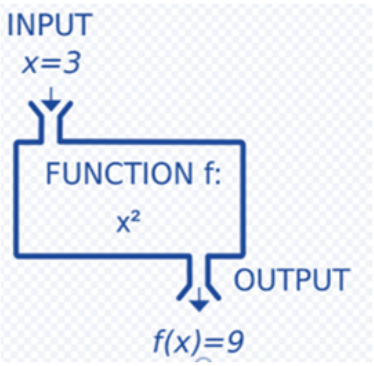
(c) This relationship is not a function because input q is associated with two different outputs.

**Functions**

A function is a relation in which each possible input value leads to exactly one output value. We say "the output is a function of the input."

The **input** values make up the **domain**, and the **output** values make up the **range**.



## Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also that we can program them into computers. There are various ways of representing functions. A standard **function notation** is one representation that makes it easier to work with functions.

To represent "height is a function of age," we start by identifying the descriptive variables $h$ for height and $a$ for age. The letters $f, g$ and $h$ are often used to represent functions just as we use $x, y$ and $z$ to represent numbers and $A, B$ and $C$ to represent sets.

$h$ is $f$ of $a$. We name the function $f$; height is a function of age.

$h = f(a)$. We use parentheses to indicate the function input.

$f(a)$. We name the function $f$; the expression is read as "$f$ of $a$".

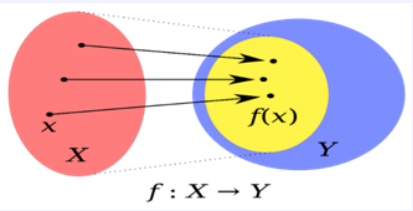
Remember, we can use any letter to name the function; we can use the notation $h(a)$ to show that $h$ depends on $a$. The input value $a$ must be put into the function $h$ to get an output value. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example $f(a + b)$ means first add $a$ and $b$ , and the result is the input for the function. We must perform the operations in this order to obtain the correct result.

## Dependent and Independent Variable

**Function Notation**

The notation $y = f(x)$ defines a function named $f$. This is read as "$y$ is a function of $x$". The letter $x$ represents the input value, or the independent variable. The letter $y$ , or $f(x)$ , represents the output value, or the dependent variable.




## Functions defined by equations

It is relatively easy to determine whether an equation is a function by solving for $y$. When you are given an equation and a specific value for $x$, there should only be one corresponding $y$-value for that $x$-value.

**Example 1**

Determine the following equations represent the function or not.

(a) $y = x + 1$

(b) $y = x^2 - 1$

(c) $y^2 = x + 5$

**Solution :**

(a) $y = x + 1$ is a function because $y$ will always be one greater than $x$. Equations with exponents can also be functions.

(b) $y = x^2 - 1$ is a function; although $x$-values of 1 and -1 give the same $y$-value (0), that is the only possible $y$-value for each of those $x$-values.

(c) $y^2 = x + 5$ is not a function; if you assume that $x = 4$, then $y^2 = 4 + 5 = 9$. $y^2 = 9$ has two possible answers 3 and -3.

### Functions defined by arrow diagrams

**Example 2**

Let $X = \{1, 2, 3\}$ and $Y = \{A, B, C, D\}$. If the arrow diagram on the right defines a function $f$ then

(a) What is the domain of $f$ ?

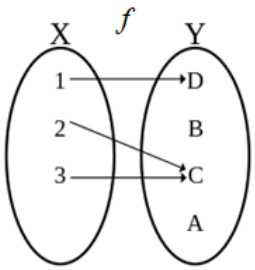
(b) What is the co-domain of $f$ ?

(c) What is $f(3)$?

(d) What is the range of $f$?

(e) What is the pre-image of $D$?

(f) What is the pre-image of $A$?




**Solution :**

(a) The domain of $f$ is $X = \{1, 2, 3\}$

(b) The co-domain of $f$ is $Y = \{A, B, C, D\}$

(c) $f(3) = C$

(d) Range of $f$ is $\{C, D\}$

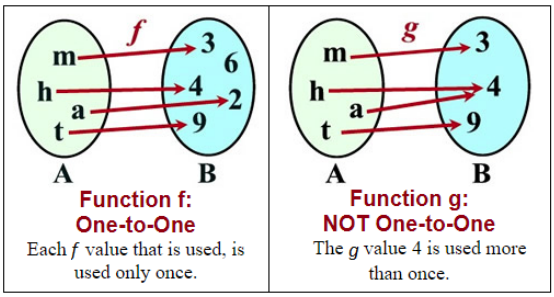
(e) The pre-image of $D$ is 1.

(f) $A$ has no pre-image.

## 10.1.1 Types of functions

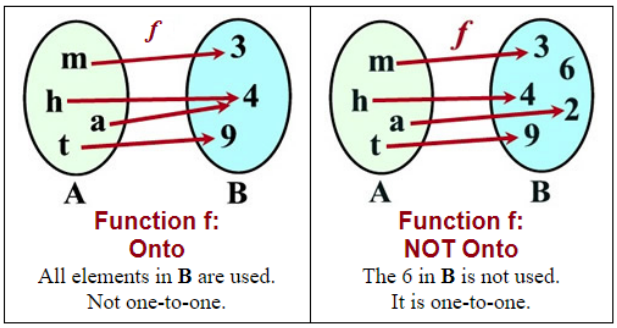
### One-to-One Function (Injective Function)

An injective function or a one-to-one function is a function $f : A \to B$ with the following property. For every element $b$ in the codomain $B$ there is maximum one element $a$ in the domain $A$ such that $f(a) = b$. For an injective function, $f(a) = f(b) \Rightarrow a = b$.



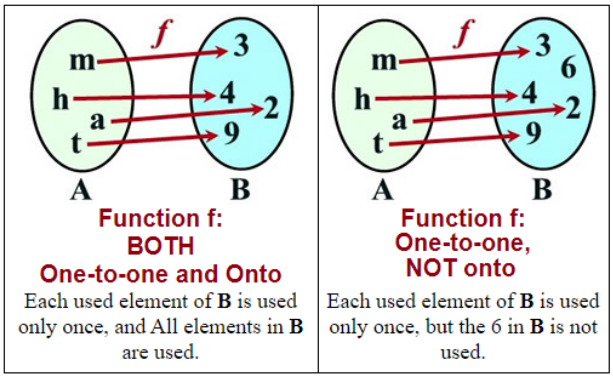

## Onto Function (Surjective Function)

A surjective function or an onto function is a function $f : A \to B$ with the following property. For every element $b$ in the codomain B there is at least one element $a$ in the domain $A$ such that $f(a) = b$. For a surjective function, the range and codomain are the same.



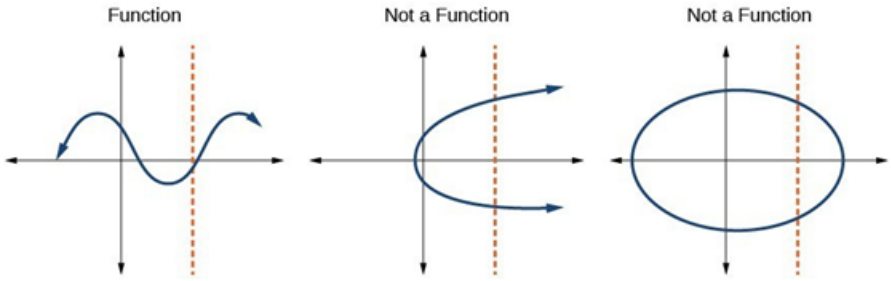

## Bijection (Bijective Function)

A bijective function is a function $f : A \to B$ that is both injective and surjective. For every element $b$ in the codomain B there is exactly one element $a$ in the domain $A$ such that $f(a) = b$.



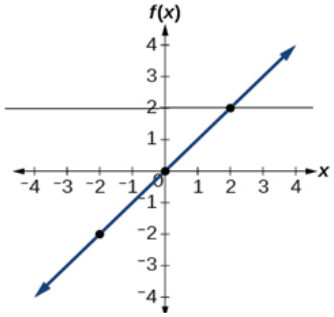

### 10.1.2 Vertical line test for a function

The vertical line test can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does not define a function because a function has only one output value for each input value.

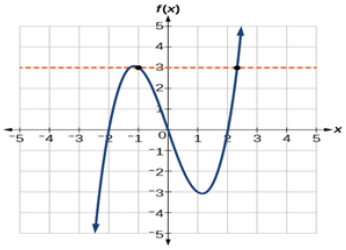



### 10.1.3 Horizontal line test for a one-to-one function

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the horizontal line test. Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.




One to One Function




Not One to One Function

### 10.1.4 Find the domain of the function

**Polynomial Function**

Let $n$ be a non-negative integer. A polynomial function is a function that can be written in the form

$$f(x) = a_n x^n + a_{n-1} n x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

This is called the general form of a polynomial function. Each $a_i$ is a coefficient and can be any real, but an cannot $a_i = 0$. Each expression $a_i x^i$ is a term of a polynomial function. The highest power of the variable that occurs in the polynomial function is called the degree of a polynomial function.

***How To* :: Find the domain of the function if function written in an equation form**

How?

Step 1. Identify the input values.

Step 2. Identify any restrictions on the input and exclude those values from the domain.

Step 3. Write the domain in interval form, if possible.

The domain of the polynomial function is the set of all real numbers denoted by $(-\infty, \infty)$ or $R$.

**Example 3**

Find the domain of the function $f(x) = x^2 - 1$

**Solution :**

The input value, shown by the variable $x$ in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form, the domain of $f$ is $(-\infty, \infty)$ or $R$ (Set of real numbers).

**Example 4**

Find the domain of the function $f(t) = 2t^7 + 2t^4 - 3t^3 + 7t^2 - 5$

**Solution :**

The input value, shown by the variable $t$ in the equation, is having different positive powers multiplied with integers and then the result is lowered by five. Any real number may be powered by positive numbers multiplied with integers and then be lowered by five, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form, the domain of $f$ is $(-\infty, \infty)$ or $R$ (Set of real numbers).

**Rational Function**

A rational function is a function that can be written as the quotient of two polynomial functions $P(x)$ and $Q(x)$.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_p x^p + a_{p-1}x^{p-1} + \ldots + a_1 x + a_0}{b_q x^q + b_{q-1}x^{q-1} + \ldots + b_1 x + b_0}; \qquad Q(x) \neq 0$$

where $p(x)$ and $q(x)$ are polynomial functions is called rational function.

***How To* ::** **Find the domain of the function if function written in an equation form that includes a fraction**

How?

Step 1. Identify the input values.

Step 2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for $x$ . If the function's formula contains an even root, set the radicand greater than or equal to 0, and then solve.

Step 3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

**Example 5**

Find the domain of the function $f(x) = \dfrac{x+1}{2-x}$

**Solution :**

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for $x$.

$$2 - x = 0$$
$$-x = -2$$
$$x = 2$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x < 2$ or $x > 2$.

In interval notation, we write the solution as: $(-\infty, 2) \cup (2, \infty)$ or $R - \{2\}$, where $R$ is the set of real numbers.

**Example 6**

Find the domain of the function $f(x) = \dfrac{3x-7}{16+x^2}$

**Solution :**

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for $x$.

$$16 + x^2 = 0$$

Now, we will not exclude any number from the domain because $16 + x^2$ is never zero for any real value of $x$. The answers are all real numbers.

In interval form, the domain of $f$ is $(-\infty, \infty)$ or $R$ (Set of real numbers).

**Example 7**

Find the domain of the function $f(x) = \dfrac{2x^3 + 3x^2}{4 - x^2}$

**Solution :**

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for $x$.

$$4 - x^2 = 0$$
$$(2 - x)(2 + x) = 0$$
$$x = -2 \text{ or } x = 2$$

Now, we will exclude $-2$ and 2 from the domain. The answers are all real numbers where $x < -2$ or $-2 < x < 2$ or $x > 2$.

In interval notation, we write the solution as: $(-\infty, 2) \cup (-2, 2) \cup (2, \infty)$ or $R - \{-2, 2\}$, where $R$ is the set of real numbers.

**Radical Function**

A function in the variable $x$ equivalent to the form

$$f(x) = \sqrt[n]{p(x)}$$

where index $n$ is the whole number and $p(x)$ is the non-negative polynomial function is called radical function

**Example 8**

Find the domain of the function $p(r) = \sqrt[3]{r + 8}$

**Solution :**

If the index of the radical function is odd number then any real number may be with no restrictions on the domain of this function. So, the domain is the set of real numbers. In interval form, the domain of $p$ is $(-\infty, \infty)$ or $R$ (Set of real numbers).

***How To* :: Find the domain of the function if function written in an equation form including an even root**




Step 1. Identify the input values.

Step 2. Since there is an even root, exclude any real numbers that result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for $x$.

Step 3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

**Example 9**

Find the domain of the function $g(t) = \sqrt{8 + t^2}$

**Solution :**

If the index of the radical function is even number and $8 + t^2$ is positive for all real $t$ then any real number may be with no restrictions on the domain of this function. So, the domain is the set of real numbers. In interval form, the domain of $g$ is $(-\infty, \infty)$ or $R$ (Set of real numbers).

**Example 10**

Find the domain of the function $f(x) = \sqrt{7 - x}$

**Solution :**

Set the radicand greater than or equal to zero and solve for $x$.

$$7 - x \geq 0$$
$$-x \geq -7$$
$$x \leq 7$$

Now, we will exclude any number greater than 7 from the domain. The answer is the set of all real numbers less than or equal to 7 or $(-\infty, 7]$

**Example 11**

Find the domain of the function $f(x) = \sqrt{9 - x^2}$

**Solution :**

Set the radicand greater than or equal to zero and solve for $x$.

$$9 - x^2 \geq 0$$
$$(3 - x)(3 + x) \geq 0$$
$$x \leq -3 \text{ and } x \geq 3$$

Now, we will exclude any number less than 3 and greater than $-3$ from the domain. The answer is the set of all real numbers less than or equal to $-3$ and greater than or equal to 3, or $(-\infty, -3] \cup [3, \infty)$

## Evaluating function at a point

**Example 12**

Let $f(t) = 4 - 5t$ and $g(x) = 3m^2 + 2m - 5$. Find the following values of the indicated functions.

(a) $f(-2)$ (b) $g(3)$ (c) $g(-3) + f(-1)$

**Solution :**

(a) $f(-2) = 4 - 5 \times (-2) = 14$

(b) $g(3) = 3 \times 3^2 + 2 \times 3 - 5 = 28$

(c) $g(-3) + f(-1)$
$= (3 \times (-3)^2 + 2 \times (-3) - 5) + 4 - 5 \times (-1)$
$= 27 - 6 - 5 + 4 + 5 = 25$

## Difference Quotient:

The difference quotient is usually the name for the expression as below:

$$\frac{f(x+h) - f(x)}{h}$$

**Example 13**

If $f(x) = 7 - 4x$ then find $\frac{f(-3+h) - f(-3)}{h}$.

**Solution :**

Given $f(x) = 7 - 4x$. Then

$$\begin{aligned} f(-3) &= 7 - 4 \times (-3) = 19 \\ f(-3+h) &= 7 - 4 \times (-3+h) \\ &= 7 + 12 - 4h \\ f(-3+h) &= 19 - 4h \end{aligned}$$

Substitute the values of $f(-3+h)$ and $f(-3)$ in the difference quotient,

$$\begin{aligned} \frac{f(-3+h) - f(-3)}{h} &= \frac{f(-3+h) - f(-3)}{h} \\ &= \frac{(19 - 4h) - (19)}{h} \\ &= \frac{-4h}{h} \\ \frac{f(-3+h) - f(-3)}{h} &= -4 \end{aligned}$$

## 10.1 EXERCISES

Answer the following questions:

I. Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. If the arrow diagram on the right defines a function $f$ then

(1) What is the domain of $f$ ?

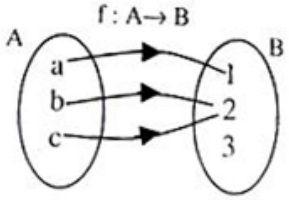
(2) What is the co-domain of $f$ ?

(3) What is $f(a)$?

(4) What is the range of $f$ ?

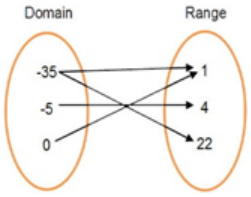
(5) What is the pre-image of 1?

(6) What is the pre-image of 3?

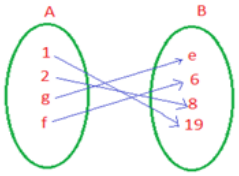



II. Which of the following are functions? If it is a function, write the domain, co-domain and range of the functions.

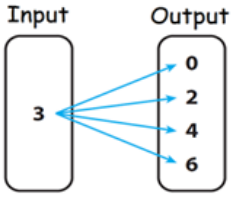
(1)



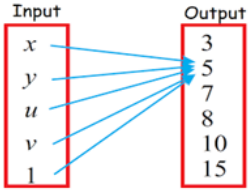

(2)



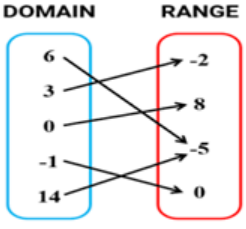

(3)



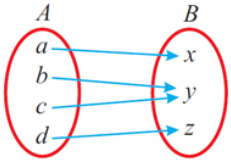

(4)




(5)




(6)



III. Which of the following are functions? If it is a function, write the domain and range for each relation.

(1) {(3, 6), (4, 9), (5, 12), (6,13)}

(2) {(2, -2), (2, -1), (2, 0), (2, 1) }

(3) {(2, 1), (3, 1), (4, 1), (5, 1)}

(4) {(10, 5), (11, 8), (12, 5), (13, 12)}

(5) {(a, -1), (b, 1), (c, 3), (d, 5)}

(6) {(2, 10), (3, 10), (4, 20), (5, 30)}

IV. Determine which of the following equations defines a function.

(1) $x + y = 7$

(2) $3x + 2y = -5$

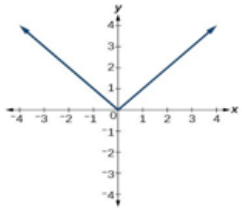
(3) $x^2 + 2y = 4$

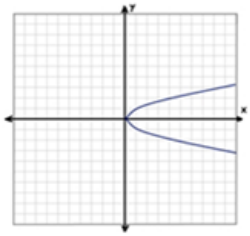
(4) $x^2 + y^2 = 9$

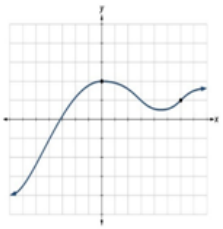
(5) $y = \sqrt{1 - x^2}$

(6) $x + y^3 = -1$

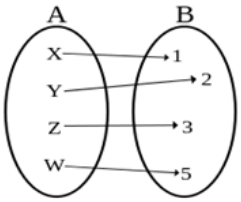
V. Identify the graph that represents a function, by using the vertical line test.

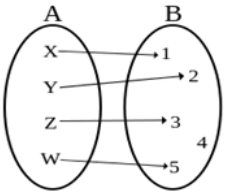
(1) 

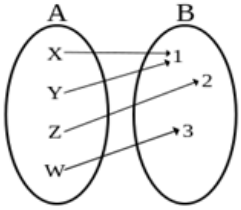
(2) 

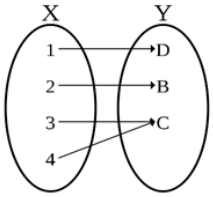
(3) 

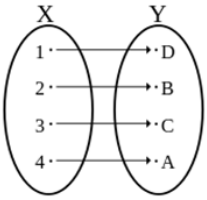
VI. Identify type of the functions in the following relations.

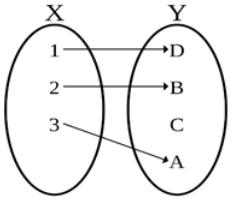
(1) 


(2) 

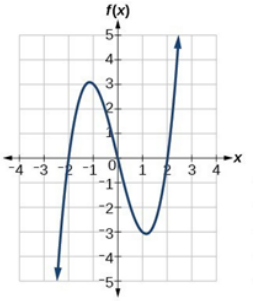

(3) 

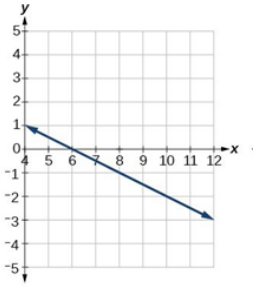

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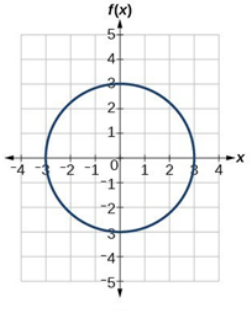

(5) 


(6) 


VII. Identify the graph that represents a one-to-one function, by using the Horizontal line test.

(1) 


(2) 


(3) 


VIII. Find the domain of the following indicated functions

(1) $f(x) = 2x^2 + 5x - 3$ (2) $f(t) = 5 - t + t^3$ (3) $f(t) = \sqrt{t-5}$

(4) $f(x) = \sqrt{5+2x}$ (5) $g(x) = \sqrt[3]{2-x}$ (6) $h(u) = \sqrt[3]{u+6}$

(7) $f(x) = \sqrt{-2+x}$ (8) $F(m) = \dfrac{3m^2+1}{m-5}$ (9) $f(x) = \dfrac{5m}{m+7}$

(10) $f(x) = \dfrac{2x}{x^2-25}$ (11) $G(m) = \dfrac{5-2m}{m^2+9}$

IX. Let $g(t) = 5-3t, f(x) = 3x+7, h(m) = 3m^2+2m-5$ and $k(u) = u^2+u-1$. Find the following values of the indicated functions.

(1) $g(-2)$ (2) $f(3)$ (3) $g(3) + f(-1)$

(4) $2h(-2) + k(5)$ (5) $f(2) - k(-3)$ (6) $\dfrac{h(2) - f(1)}{g(2)}$

X. Answer the following:

(1) If $F(s) = 2s + 3$ then find $\dfrac{F(2+h) - F(2)}{h}$

(2) If $f(x) = 2 - 5x$ then find $\dfrac{f(3+h) - f(3)}{h}$

(3) If $f(x) = 10 - 3x$ then find $\dfrac{f(-2+h) - f(-2)}{h}$

XI. Find and simplify i) $\dfrac{f(x+h) - f(x)}{h}$ ii) $\dfrac{f(x) - f(a)}{x-a}$ for the following functions

(1) $f(x) = 4x + 1$ (2) $f(x) = 3 - 7x$ (3) $f(x) = -2x^2 + 3x - 1$

## 10.2 Composition of Functions

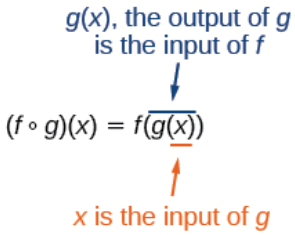
Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of combining functions so that the output of one function becomes the input of another is known as a composition of functions. The resulting function is known as a composite function. We represent this combination by the following notation:

$$(f \circ g)(x) = f(g(x))$$

We read the left-hand side as "$f$ composed with $g$ at $x$" and the right-hand side as "$f$ of $g$ of $x$". The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value.

**Composition** is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function $g$ takes the input $x$ first and yields an output $g(x)$. Then the function $f$ takes $g(x)$ as an input and yields an output $f(g(x))$.




In general, $f \circ g$ and $g \circ f$ are different functions. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all $x$. We will also see that sometimes two functions can be composed only in one specific order.

**Example 1**

If $f(x) = x^2$ and $g(x) = x + 2$, then find $f(g(x))$ and $g(f(x))$.

**Solution :**

Given $f(x) = x^2$ and $g(x) = x + 2$,

$$\begin{aligned} f(g(x)) &= f(x+2) \\ &= (x+2)^2 \\ &= x^2 + 4x + 4 \end{aligned} \qquad \begin{aligned} g(f(x)) &= g\left(x^2\right) \\ &= x^2 + 2 \end{aligned}$$

These expressions are not equal for all values of $x$, so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value $x = -\frac{1}{2}$.

**Example 2**

If $f(x) = 2x + 1$ and $g(x) = 3 - x$, then find $(f \circ g)(x)$ and $(g \circ f)(x)$.

**Solution :**

Given $f(x) = 2x + 1$ and $g(x) = 3 - x$,

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(3 - x) \\ &= 2(3 - x) + 1 \\ &= 6 - 2x + 1 \\ &= 7 - 2x\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(2x + 1) \\ &= 3 - (2x + 1) \\ &= 3 - 2x - 1 \\ &= 2 - 2x\end{aligned}$$

## 10.2 EXERCISES

I. Find $f\,(g\,(x))$ and $g\,(f\,(x))$ for the following functions.

(1) $f(x) = 3x + 2$ and $g\,(x) = x + 1$

(2) $f(x) = x + 2$ and $g\,(x) = x^2$

(3) $f(x) = x^2 + 1$ and $g\,(x) = \sqrt{x + 2}$

(4) $f(x) = \sqrt{x} + 2$ and $g\,(x) = x^2 + 3$

(5) $f(x) = |x|$ and $g\,(x) = 5x + 1$

II. Evaluate each composite value.

(1) $f(x) = 2x + 5$ and $g\,(x) = x - 2$, find $(f \circ g)\,(2)$

(2) $f(x) = x + 1$ and $g\,(x) = \sqrt{x - 7}$, find $(f \circ g)\,(16)$

(3) $f(x) = x + 5$ and $g\,(x) = \dfrac{1}{x - 3}$, find $(f\,o\,g)\,(2)$ and $(g \circ f)\,(1)$

(4) $f(x) = x + 3$ and $g\,(x) = \dfrac{x}{x - 4}$, find $(f \circ g)\,(6)$ and $(g \circ f)\,(2)$

(5) $f(x) = \dfrac{1 - x}{x}$ and $g\,(x) = \dfrac{1}{1 + x^2}$, find $(g \circ f)\,(2)$

# 10.3 Inverse Function

An inverse function reverses the operation done by a particular function. In other words, whatever a function does, the inverse function undoes it. In this section, we define an inverse function formally and state the necessary conditions for an inverse function to exist. We examine how to find an inverse function.

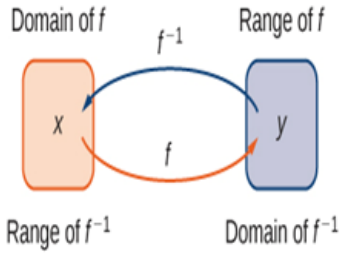
### Existence of an Inverse Function

We begin with an example. Given a function $f$ and an output $y = f(x)$, we are often interested in finding what value or values of $x$ were mapped to $y$ by $f$. For example, consider the function $f(x) = x^3 + 4$. Since any output is $f(x) = x^3 + 4$, we can solve this equation for $x$ to find that the input is $x = \sqrt[3]{y-4}$. This equation defines $x$ as a function of $y$. Denoting this function as $f^{-1}$, and writing $x = f^{-1}(y)$, we see that for any $x$ in the domain of $f$, $f^{-1}(f(x)) = f^{-1}(x^3+4) = x$. A function with this property is called the inverse function of the original function.

**Inverse Function**

Given a function $f$ with domain $\boldsymbol{D}$ and range $\boldsymbol{R}$, its ***inverse function*** (if it exists) is the function $f^{-1}$ with domain $\boldsymbol{R}$ and range $\boldsymbol{D}$ such that $f^{-1}(y) = x$ if $f(x) = y$. In other words, for a function $f$ and its inverse $f^{-1}$,
$f^{-1}(f(x)) = x$, for all $x$ in $\boldsymbol{D}$ and $f\left(f^{-1}(y)\right) = y$, for all $y$ in $\boldsymbol{R}$

Recall that a function has exactly one output for each input. Therefore, to define an inverse function, we need to map each input to exactly one output. For example, let's try to find the inverse function for $f(x) = x^2$. Solving the equation $f(x) = x^2$, we arrive at the equation $x = \pm\sqrt{y}$. This equation does not describe $x$ as a function of $y$ because there are two solutions to this equation for every $y > 0$. The problem with trying to find an inverse function for $f(x) = x^2$ is that two inputs are sent to the same output for each output $y > 0$. The function $f(x) = x^3 + 4$ discussed earlier did not have this problem. For that function, each input was sent to a different output. A function that sends each input to a different output is called a one-to-one function.




***How To*** **:: Find a Function's Inverse**

How?

Step 1. Make sure $f$ is one-to-one function. If $f$ is not one-to-one, then $f^{-1}$ does not exist.

Step 2. Solve for $x$ and write $x = f^{-1}(y)$.

Step 3. Interchange $x$ and $y$.

Step 4. Find the domain of $f^{-1}$.The domain of $f^{-1}$ must be the same as the range of $f$.

**Example 1**

Find the inverse of the function $f(x) = 3x - 4$.

**Solution :**

Let $y = f(x)$,

| | | |
|---|---|---|
| $y$ | $= 3x - 4$ | Set up an equation. |
| $y + 4$ | $= 3x$ | Add 4 both sides. |
| $x$ | $= \frac{y+4}{3}$ | divide both sides by 3 |
| $f^{-1}(y)$ | $= \frac{y+4}{3}$ | Replace $x = f^{-1}(y)$ |
| $f^{-1}(x)$ | $= \frac{x+4}{3}$ | Replace $y$ with $x$ |

Since the domain of $f$ is $(-\infty, \infty)$, the range of $f^{-1}$ is $(-\infty, \infty)$.

Since the range of $f$ is $(-\infty, \infty)$, the domain of $f^{-1}$ is $(-\infty, \infty)$.

**Example 2**

Find the inverse of the function $f(x) = \frac{2}{x-3} + 4$.

**Solution :**

Let $y = f(x)$,

| | | |
|---|---|---|
| $y$ | $= \frac{2}{x-3} + 4$ | Set up an equation. |
| $y - 4$ | $= \frac{2}{x-3}$ | Subtract 4 from both sides. |
| $x - 3$ | $= \frac{2}{y-4}$ | Multiply both sides by $x-3$ and divide by $y-4$ |
| $x$ | $= \frac{2}{y-4} + 3$ | Add 3 to both sides. |
| $f^{-1}(y)$ | $= \frac{2}{y-4} + 3$ | Replace $x = f^{-1}(y)$ |
| $f^{-1}(x)$ | $= \frac{2}{x-4} + 3$ | Replace $y$ with $x$ |

Since the domain of $f$ is $R - \{3\}$, the range of $f^{-1}$ is $R - \{3\}$.

Since the range of $f$ is $R - \{4\}$, the domain of $f^{-1}$ is $R - \{4\}$.

## 10.3 EXERCISES

I. Find the inverse function $f^{-1}$ if exist.

(1) $f = \{(3,6),(4,9),(5,12),(6,13)\}$

(2) $f = \{(2,-2),(4,-1),(6,0),(8,1)\}$

(3) $f = \{(2,1),(3,1),(4,1),(5,1)\}$

(4) $f = \{(10,5),(11,8),(12,5),(13,12)\}$

(5) $f = \{(a,-1),(b,1),(c,3),(d,5)\}$

(6) $f = \{(2,10),(3,10),(4,20),(5,30)\}$

II. Find the inverse function $f^{-1}(x)$.

(1) $f(x) = x + 3$

(2) $f(x) = 2 - x$

(3) $f(x) = 5x - 1$

(4) $f(x) = \frac{1}{7}x + 5$

(5) $f(x) = \frac{x-4}{3}$

(6) $f(x) = -\frac{2x-3}{5}$

(7) $f(x) = \frac{5}{8+x} - 6$

(8) $f(x) = \frac{2x+3}{5x+4}$

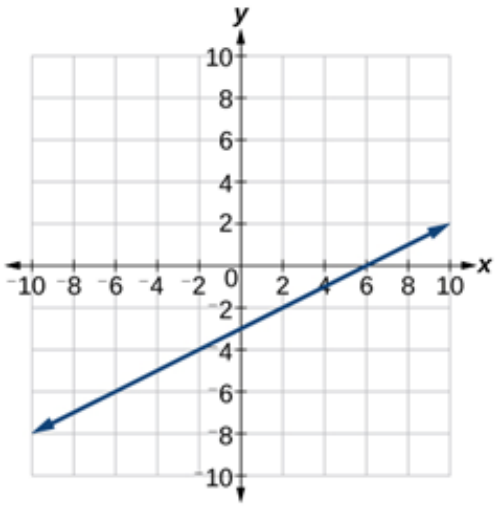
(9) $f(x) = 2 + \sqrt{x-4}$

## 10.4 Graphing Functions

### 10.4.1 Representing a Linear Function in Graphical Form

In Linear Functions, we saw that that the graph of a linear function is a straight line. We were also able to see the points of the function as well as the initial value from a graph. By graphing two functions, then, we can more easily compare their characteristics.

The function describing the train's motion is a linear function, which is defined as a function with a constant rate of change, that is, a polynomial of degree 1. There are several ways to represent a linear function, including word form, function notation, tabular form, and graphical form. We will describe the train's motion as a function using each method.



**Linear Function**

A linear function is a function whose graph is a line. Linear functions can be written in the slope-intercept form of a line.

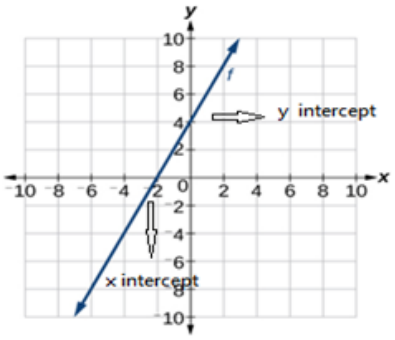
$$f(x) = mx + b$$

where $b$ is the initial or starting value of the function (when input,$x = 0$ ), and $m$ is the constant rate of change, or slope of the function. The $y$-intercept is at $(0, b)$.

**Finding the $x-$intercept of a Line:**

To find the $x$-intercept, set a function equal to zero and solve for the value of $x$.

$$\begin{aligned} f(x) &= 0 \\ mx + b &= 0 \\ mx &= -b \\ x &= \frac{-b}{m} \\ \therefore\ x - \text{intercept} &= -\frac{b}{m} \end{aligned}$$



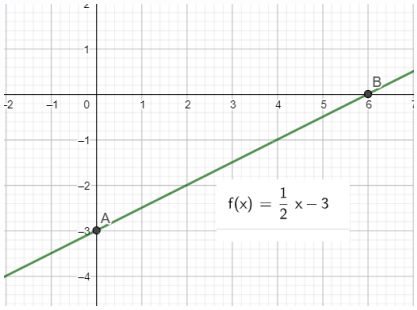

**Example 1**

Find the slope and intercepts, and then sketch the graph of the linear function defined by $f(x) = \frac{1}{2}x - 3$.

**Solution :**

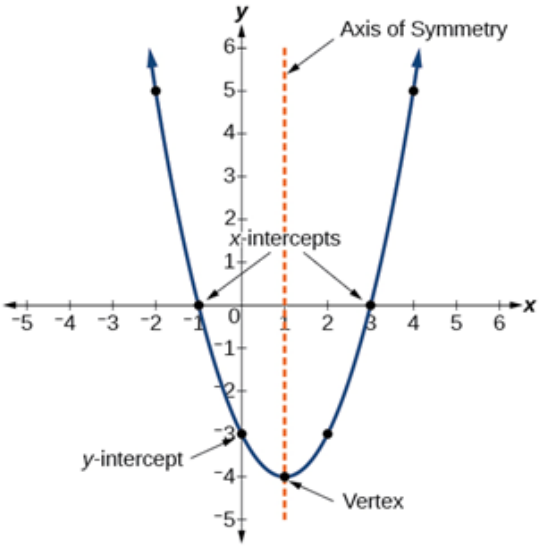
Given

$$\begin{aligned} f(x) &= \frac{1}{2}x - 3 \\ f(x) &= mx + b \\ \text{slope } (m) &= \frac{1}{2} \text{ and } b = -3 \\ x - \text{intercept} &= \frac{-b}{m} = \frac{3}{1/2} = 6 \end{aligned}$$



### 10.4.2 Representing a Quadratic Function in Graphical Form

The graph of a quadratic function is a U-shaped curve called a **parabola**. One important feature of the graph is that it has an extreme point, called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the minimum value of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the maximum value. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**.




The $y$-intercept is the point at which the parabola crosses the $y$-axis. The $x$-intercepts are the points at which the parabola crosses the $x$-axis. If they exist, the $x$-intercepts represent the **zeros**, or **roots**, of the quadratic function, the values of $x$ at which $y = 0$.

**Quadratic Functions**

A quadratic function is a function of degree two. The graph of a quadratic function is a **parabola**. The general form of a quadratic function is $f(x) = ax^2 + bx + c$ where $a, b$ and $c$ are real numbers and $a \neq 0$.

The standard form of a quadratic function is $f(x) = a(x - h)^2 + k$.

The vertex $(h, k)$ is located as $h = -\dfrac{b}{2a}$ and $k = f(h)$

**Properties:**

1. If $a > 0$ the parabola opens up and if $a < 0$ the parabola opens down.
2. Determine whether $a$ is positive or negative. If $a$ is positive, the parabola has a minimum. If $a$ is negative, the parabola has a maximum.
3. Determine the maximum or minimum value of the parabola,$k$.
4. If the parabola has a minimum, the range is given by $f(x) \geq k$ or $[k, \infty)$. If the parabola has a maximum, the range is given by $f(x) \leq k$ or $(-\infty, k]$.
5. If $a > 0$, then the graph decreases in the interval $(-\infty, h]$ and increase in the interval $[h, \infty)$.
6. If $a < 0$, then the graph increases in the interval $(-\infty, h]$ and decrease in the interval $[h, \infty)$.

### 10.4.3 Finding the Vertex of a Quadratic Function

***How To* :: Find the vertex of the parabola for a quadratic function in general form,**




Step 1. Identify $a, b$, and $c$.

Step 2. Find $h$, the $x-$coordinate of the vertex, by substituting $a$ and $b$ into $h = \frac{-b}{2a}$.

Step 3. Find $k$, the $y-$coordinate of the vertex, by evaluating

$$k = f(h) = f\left(\frac{-b}{2a}\right)$$

**Example 2**

Find the vertex of the quadratic function $f(x) = 2x^2-6x+7$. Rewrite the quadratic in standard form (vertex form).

**Solution :**

The horizontal coordinate of the vertex

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{-6}{2(2)} \\ &= \frac{6}{4} \\ &= \frac{3}{2} \end{aligned}$$

The vertical coordinate of the vertex

$$\begin{aligned} k &= f(h) \\ &= f\left(\frac{3}{2}\right) \\ &= 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 \\ &= \frac{5}{2} \end{aligned}$$

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic. First, find the horizontal coordinate of the vertex. Then find the vertical coordinate of the vertex. Substitute the values into standard form, using the a from the general form.

$$f(x) = ax^2 + bx + c$$
$$f(x) = 2x^2-6x + 7$$

The standard form of a quadratic function prior to writing the function then becomes the following:

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2}$$

### 10.4.4 Finding the Domain and Range of a Quadratic Function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is all real numbers. Because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all $y-$values greater than or equal to the $y-$coordinate at the turning point or less than or equal to the $y-$coordinate at the turning point, depending on whether the parabola opens up or down.

***How To* :: Find the domain and range**




Step 1. Identify the domain of any quadratic function as all real numbers.

Step 2. Determine whether a is positive or negative. If a is positive, the parabola has a minimum. If a is negative, the parabola has a maximum.

Step 3. Determine the maximum or minimum value of the parabola, $k$.

Step 4. If the parabola has a minimum, the range is given by $f(x) \geq k$, or $[k, \infty)$. If the parabola has a maximum, the range is given by $f(x) \leq k$, or $(-\infty, k]$.

**Example 3**

Find the domain and range of $f(x) = -5x^2 + 9x - 1$.

**Solution :**

As with any quadratic function, the domain is all real numbers.

Because $a$ is negative, the parabola opens downward and has a maximum value. We need to determine the maximum value. We can begin by finding the $x-$value of the vertex.

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{9}{2(-5)} \\ &= \frac{9}{10} \end{aligned}$$

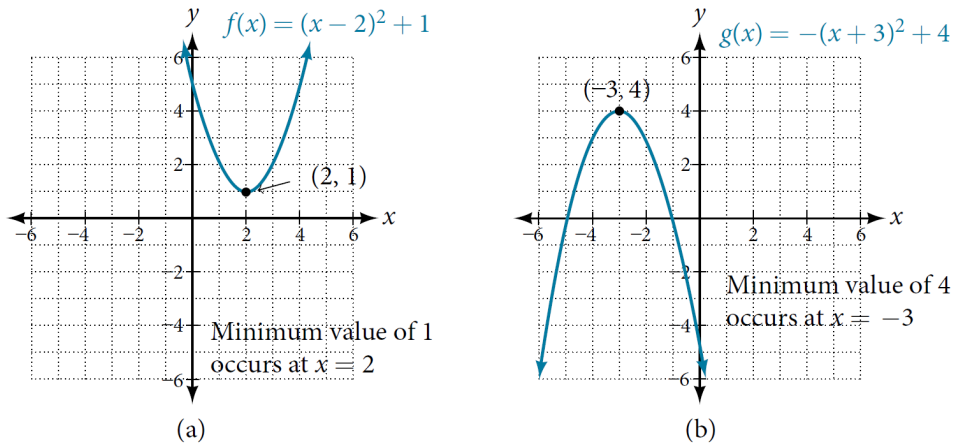
The maximum value is given by $f(h)$.

$$\begin{aligned} f\left(\frac{9}{10}\right) &= -5\left(\frac{9}{10}\right)^2 + 9\left(\frac{9}{10}\right) - 1 \\ &= \frac{61}{20} \end{aligned}$$

The range is $f(x) \leq \frac{61}{20}$, or $\left(-\infty, \frac{61}{20}\right]$

### 10.4.5 Determining the Maximum and Minimum Values of Quadratic Functions

The output of the quadratic function at the vertex is the maximum or minimum value of the function, depending on the orientation of the parabola. We can see the maximum and minimum values in the below images.




There are many real-world scenarios that involve finding the maximum or minimum value of a quadratic function, such as applications involving area and revenue.

**Example 4**

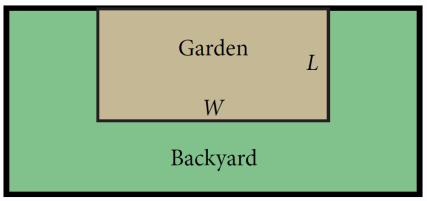
A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

(a) Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.

(b) What dimensions should she make her garden to maximize the enclosed area?

**Solution :**

Lets use a diagram to record the given information. It is also helpful to introduce a temporary variable, W, to represent the width of the garden and the length of the fence section parallel to the backyard fence.




(a) We have only 80 feet of fence available, and $L + W + L = 80$, or more simply, $2L + W = 80$. This allows us to represent the width, $W$, in terms of $L$.

$$W = 80 - 2L$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$$A = LW = L(80 - 2L)$$
$$A(L) = 80L - 2L^2$$

This formula represents the area of the fence in terms of the variable length $L$. The function, written in general form, is

$$A(L) = -2L^2 + 80L$$

(b) The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value for the area. In finding the vertex, we must be careful because

the equation is not written in standard polynomial form with decreasing powers. This is why we rewrote the function in general form above.

Since a is the coefficient of the squared term, $a = -2, b = 80$, and $c = 0$.

To find the vertex:

$$h = -\frac{b}{2a}$$
$$h = -\frac{80}{2(-2)}$$
$$= 20$$

$$k = A(20)$$
$$= 80(20) - 2(20)^2$$
$$= 800$$

The maximum value of the function is an area of 800 square feet, which occurs when $L = 20$ feet. When the shorter sides are 20 feet, there is 40 feet of fencing left for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet and the longer side parallel to the existing fence has length 40 feet.

**Example 5**

The unit price of an item affects its supply and demand. That is, if the unit price goes up, the demand for the item will usually decrease. For example, a local newspaper currently has 84,000 subscribers at a quarterly charge of \$30. Market research has suggested that if the owners raise the price to \$32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

**Solution :**

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the price per subscription times the number of subscribers, or quantity. We can introduce variables, $p$ for price per subscription and $Q$ for quantity, giving us the equation Revenue $= pQ$.

Because the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $p = 30$ and $Q = 84,000$. We also know that if the price rises to \$32, the newspaper would lose 5,000 subscribers, giving a second pair of values, $p = 32$ and $Q = 79,000$. From this we can find a linear equation relating the two quantities. The slope will be

$$m = \frac{79,000 - 84,000}{32 - 30}$$
$$= \frac{-5,000}{2}$$
$$= -2,500$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the $y-$intercept.

$$Q = -2,500p + b \qquad \text{Substitute in the point } Q = 84,000 \text{ and } p = 30$$
$$84,000 = -2,500(30) + b \qquad \text{Solve for } b$$
$$b = 159,000$$

This gives us the linear equation $Q = -2,500p + 159,000$ relating cost and subscribers. We now return to our revenue equation.

$$\text{Revenue} = pQ$$
$$\text{Revenue} = p(-2,500p + 159,000)$$
$$\text{Revenue} = -2,500p^2 + 159,000p$$

We now have a quadratic function for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex.

$$h = -\frac{159,000}{2(-2,500)}$$

$$= 31.8$$

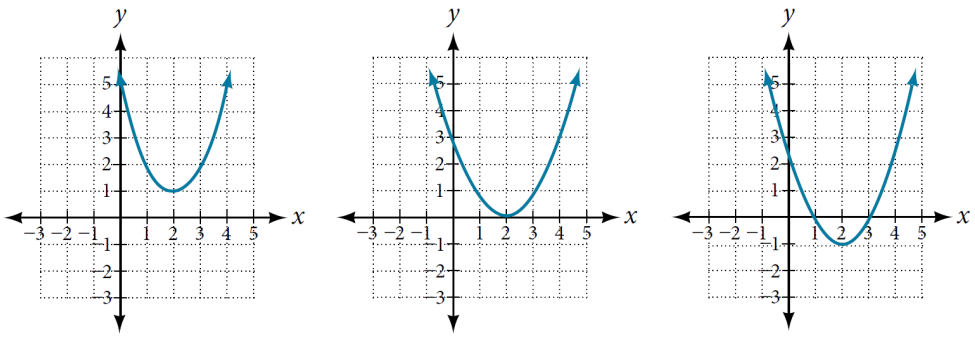
The model tells us that the maximum revenue will occur if the newspaper charges \$31.80 for a subscription. To find what the maximum revenue is, we evaluate the revenue function.

$$\text{Maximum Revenue } = -2,500(31.8)2 + 159,000(31.8)$$

$$= 2,528,100$$

### 10.4.6 Finding the $x-$and $y-$Intercepts of a Quadratic Function

Much as we did in the application problems above, we also need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the $y-$intercept of a quadratic by evaluating the function at an input of zero, and we find the $x-$intercepts at locations where the output is zero. Notice that the number of $x-$intercepts can vary depending upon the location of the graph.




***How To* :: Find the $y-$and $x-$intercepts.**

How?

Step 1. Evaluate $f(0)$ to find the $y-$intercept.

Step 2. Solve the quadratic equation $f(x) = 0$ to find the $x-$intercepts.

**Example 6**

Find the $y-$intercept and $x-$intercept of the quadratic $f(x) = 3x^2 + 5x-2$.

**Solution :**

We find the $y-$intercept by evaluating f (0).

$$f(0) = 3(0)^2 + 5(0)-2$$

$$= -2$$

So the $y-$intercept is at $(0, -2)$.

For the $x-$intercepts, we find all solutions of $f(x) = 0$.

$$0 = 3x^2 + 5x-2$$

In this case, the quadratic can be factored easily, providing the simplest method for solution.

$$0 = (3x-1)(x+2)$$

$$0 = 3x-1 or 0 = x+2$$

$$x = \frac{1}{3} \text{ or } x = -2$$

So the $x-$intercepts are at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$.

**Example 7**

Find the $y-$intercept and $x-$intercept of the quadratic $f(x) = -2x^2 - 2x + 4$.

**Solution :**

We find the $y-$intercept by evaluating f $(0)$.

$$f(0) = -2(0)^2 - 2(0) + 4$$

$$= 4$$

So the $y-$intercept is at $(0, 4)$.

For the $x-$intercepts, we find all solutions of $f(x) = 0$.

$$0 = -2x^2 - 2x + 4$$

In this case, the quadratic can be factored easily, providing the simplest method for solution.

$$0 = -2(x-1)(x+2)$$

$$0 = x-1 or 0 = x+2$$

$$x = 1 \text{ or } x = -2$$

So the $x-$intercepts are at $(1, 0)$ and $(-2, 0)$.

### 10.4.7 Sketch the Graph of the Quadratic Function

**Example 8**

Sketch the graph of the following quadratic function $f(x) = x^2 + 4x + 3$ and find the following:

(a) Vertex
(b) Axis of symmetry
(c) Range
(d) Maximum
(e) Minimum
(f) Interval of decreasing
(g) Interval of increasing
(h) $x-$ intercept
(i) $y-$ intercept

**Solution :**

Given

$$f(x) = x^2 + 4x + 3$$

$$f(x) = ax^2 + bx + c$$

$$a = 1, b = 4, \text{ and } c = 3$$

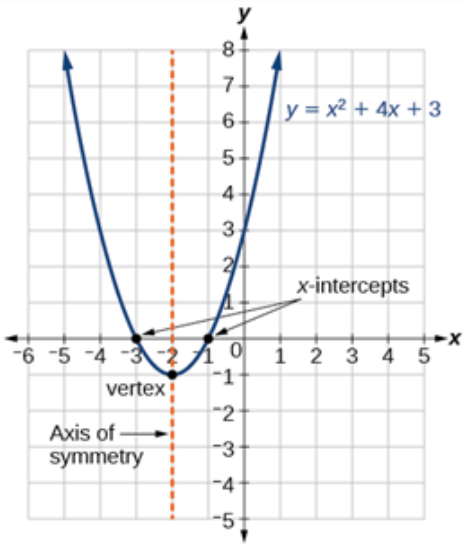
$$h = -\frac{b}{2a} = -\frac{4}{2} = -2$$

$$k = f(h)$$

$$= f(-2)$$

$$= (-2)^2 + 4(-2) + 3$$

$$= -1$$




$a > 0$ indicates that the parabola opens upward.

| $x$ | -4 | -3 | -2 | -1 | 0 |
|---|---|---|---|---|---|
| $f(x)$ | 3 | 0 | -1 | 0 | 3 |

(a) vertex : $(h, k) = (-2, -1)$

(b) Axis of symmetry: $x = -2$

(c) Range: $[k, \infty) = [-1, \infty)$

(d) Maximum: None

(e) Minimum: $k = -1$

(f) Interval of decreasing: $(-\infty, h] = (-\infty, -2]$

(g) Interval of increasing : $[h, \infty) = [-2, \infty)$

(h) $x-$ intercepts are $(-3, 0)$ and $(-1, 0)$.
For,

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(3)}}{2(2)}$$

(i) $y-$ intercept is $(0, 3)$

**Example 9**

Sketch the graph of the following quadratic function $f(x) = -3(x + 2)^2 + 4$ and find the following:

(a) Vertex

(b) Axis of symmetry

(c) Range

(d) Maximum

(e) Minimum

(f) Interval of decreasing

(g) Interval of increasing

(h) $x-$ intercept

(i) $y-$ intercept

**Solution :**

Given

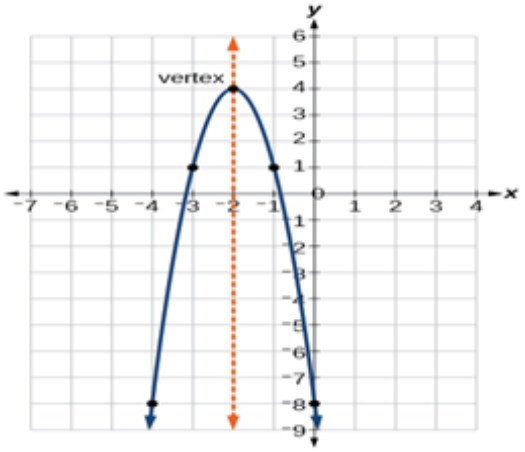
$$f(x) = -3(x+2)^2 + 4$$

$$f(x) = a(x-h)^2 + k$$

$$a = -3, \quad h = -2, \quad k = 4$$

$a < 0$ indicates that the parabola opens downward.

| $x$ | -4 | -3 | -2 | -1 | 0 |
|---|---|---|---|---|---|
| $f(x)$ | -8 | 1 | 4 | 1 | -8 |




(a) vertex : $(h, k) = (-2, 4)$

(b) Axis of symmetry: $x = -2$

(c) Range: $(-\infty, k] = (-\infty, 4]$

(d) Maximum: $k = 4$

(e) Minimum: None

(f) Interval of increasing: $(-\infty, h] = (-\infty, -2]$

(g) Interval of decreasing : $[h, \infty) = [-2, \infty)$

(h) $x-$ intercepts are $(-3.33, 0)$ and $(-1.33, 0)$.
For,

$$x = -2 \pm \sqrt{\frac{4}{3}}$$

(i) $y-$ intercept is $(0, -8)$.
For,

$$f(0) = -3(0+2)^2 = -8$$

## 10.4 EXERCISES

Answer the following questions:

I. Find the slope and intercepts and then sketch the graph of the linear functions.

(1) $f(x) = 3x - 6$

(2) $f(x) = -2x + 3$

(3) $f(x) = 4 + 2x$

(4) $f(x) = \frac{1}{4}x - 4$

(5) $f(x) : -3x + 2y = 12$

II. Sketch the graph of the quadratic functions and find the following:

(a) Vertex
(b) Axis of symmetry
(c) Range
(d) Maximum
(e) Minimum
(f) Interval of decreasing
(g) Interval of increasing
(h) $x-$ intercept
(i) $y-$ intercept

(1) $f(x) = 2x^2 - 4x + 2$

(2) $f(x) = -x^2 + 4x + 3$

(3) $f(x) = 3x^2 - 6x - 9$

(4) $f(x) = (x - 3)^2 + 2$

(5) $f(x) = -2(x + 3)^2 - 6$

(6) $f(x) = \frac{1}{2}(x + 2)^2 - 3$

III. Real-World Applications: Determining the Maximum and Minimum Values of Quadratic Functions

(1) Find the dimensions of the rectangular corral producing the greatest enclosed area given 200feet of fencing.

(2) Find the dimensions of the rectangular corral split into 2 pens of the same size producing the greatest possible enclosed area given 300 feet of fencing.

(3) Among all of the pairs of numbers whose sum is 6, find the pair with the largest product. What is the product?

(4) Among all of the pairs of numbers whose difference is 12, fi d the pair with the smallest product. What is the product?

(5) A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 229t + 234$. Find the maximum height the rocket attains.

(6) A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 24t + 8$. How long does it take to reach maximum height?

# Chapter 11

# Exponential and Logarithmic Functions

## Contents


**11.1 Exponential and Logarithmic Functions:**

11.1.1 Graphing Exponential Functions

11.1.2 Properties of Exponential Functions

11.1.3 Graphing Logarithmic Functions

11.1.4 Properties of Logarithmic Functions

**11.2 Applications of Exponential and Logarithmic Functions**


**Learning outcomes covered:**

(s) `Define and manipulate exponential and logarithmic functions and solve problems arising from real life applications.`

(t) `Understand the inverse relationship between exponents and logarithms and use this relationship to solve related problems.`

### Learning Objectives

**By the end of this chapter, the students will be able to:**

- Understand the concept of exponential and logarithmic functions.
- Understand the properties of exponential and logarithmic functions.
- Solve problems based on exponential and logarithmic equations.

# Introduction

In this chapter, it is introduced the properties of exponential and logarithmic functions and the method of applying these properties in solving problems. At the end, some applications are given using exponential and logarithmic functions.

## 11.1 Exponential and Logarithmic Functions:

**Exponential Function**

The equation

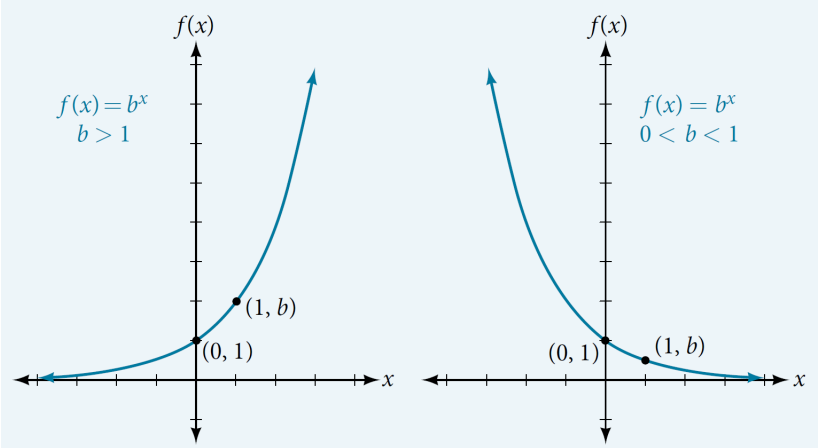
$$f(x) = b^x, \qquad b > 0, \qquad b \neq 1$$

defines an exponential function for each different constant $b$, called the **base**. The independent variable $x$ can assume any real value.

### Characteristics of the graph of the parent function $f(x) = b^x$

An exponential function with the form $f(x) = b^x, b > 0, b \neq 1$, has these characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- $x$ -intercept: none
- $y$ -intercept: $(0, 1)$
- increasing if $b > 1$
- decreasing if $b < 1$




The figure compares the graphs of exponential growth and decay functions.

### 11.1.1 Graphing Exponential Functions

***How To* :: Graph the exponential function of the form $f(x) = b^x$**




Step 1. Create a table of points.

Step 2. Plot at least 3 point from the table, including the $y-$intercept $(0, 1)$.

Step 3. Draw a smooth curve through the points.

Step 4. State the domain, $(-\infty, \infty)$, the range, $(0, \infty)$, and the horizontal asymptote, $y = 0$.

**Example 1**

Sketch a graph of $f(x) = 0.25^x$. State the domain, range, and asymptote.

**Solution :**

Before graphing, identify the behavior and create a table of points for the graph.

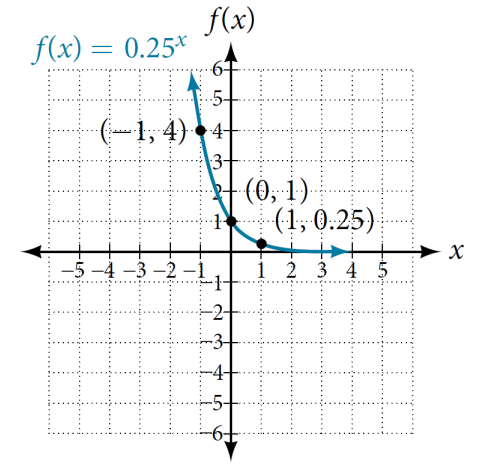
Since $b = 0.25$ is between zero and one, we know the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.

Plot the $y-$intercept, $(0, 1)$, along with two other points. We can use $(-1, 4)$ and $(1, 0.25)$.

Draw a smooth curve connecting the points as in the figure. The domain, $(-\infty, \infty)$, the range, $(0, \infty)$, and the horizontal asymptote, $y = 0$.

Create a table of points as in the table below.

| $\boldsymbol{x}$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $\boldsymbol{f(x)}$ | $64$ | $16$ | $4$ | $1$ | $0.25$ | $0.06$ | $0.02$ |




**Shifts of the parent function $f(x) = b^x$**

For any constants $c$ and $d$, the function $f(x) = b^{x+c} + d$ shifts the parent function $f(x) = b^x$

- vertically $d$ units, in the same direction of the sign of $d$.
- horizontally $c$ units, in the opposite direction of the sign of $c$.
- The $y$ -intercept becomes $(0, b^c + d)$.
- The horizontal asymptote becomes $y = d$.
- The range becomes $(d, \infty)$.
- The domain, $(-\infty, \infty)$, remains unchanged.

**Stretches and compressions of the parent function $f(x) = b^x$**

For any factor $a > 0$, the function $f(x) = a(b)^x$

- is stretched vertically by a factor of $a$ if $|a| > 1$.
- is compressed vertically by a factor of $a$ if $|a| < 1$
- has a $y$ -intercept of $(0, a)$.
- has a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$, which are unchanged from the parent function.

**Example 2**

Graph $f(x) = 2^{x+1} - 3$. State the domain, range, and asymptote.

**Solution :**

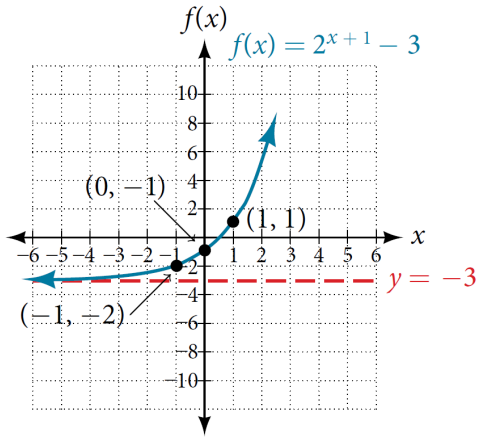
We have an exponential equation of the form $f(x) = b^{x+c} + d$, with $b = 2, c = 1$, and $d = -3$.

Draw the horizontal asymptote $y = d$, so draw $y = -3$.

Identify the shift as $(-c, d)$, so the shift is $(-1, -3)$.

Shift the graph of $f(x) = b^x$ left 1 units and down 3 units.

The domain, $(-\infty, \infty)$, the range, $(0, \infty)$, and the horizontal asymptote, $y = 0$.




**Example 3**

Sketch a graph of $f(x) = 4\left(\frac{1}{2}\right)^x$. State the domain, range, and asymptote.

**Solution :**

Before graphing, identify the behavior and create a table of points for the graph.

Since $b = \frac{1}{2}$ is between zero and one, the left tail of the graph will increase without bound as x decreases, and the right tail will approach the $x-$axis as $x$ increases. Since $a = 4$, the graph of $f(x) = 4\left(\frac{1}{2}\right)^x$ will be stretched by a factor of 4.

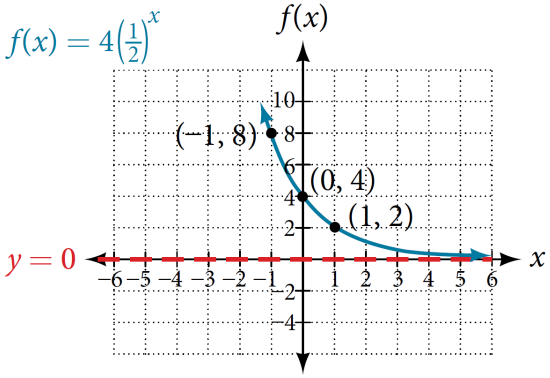
Plot the $y-$intercept, $(0, 4)$, along with two other points. We can use $(-1, 8)$ and $(1, 2)$.

Draw a smooth curve connecting the points as in the figure.

The domain, $(-\infty, \infty)$, the range, $(0, \infty)$, and the horizontal asymptote, $y = 0$.

Create a table of points as in the table below.

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 32 | 16 | 8 | 4 | 2 | 1 | 0.5 |



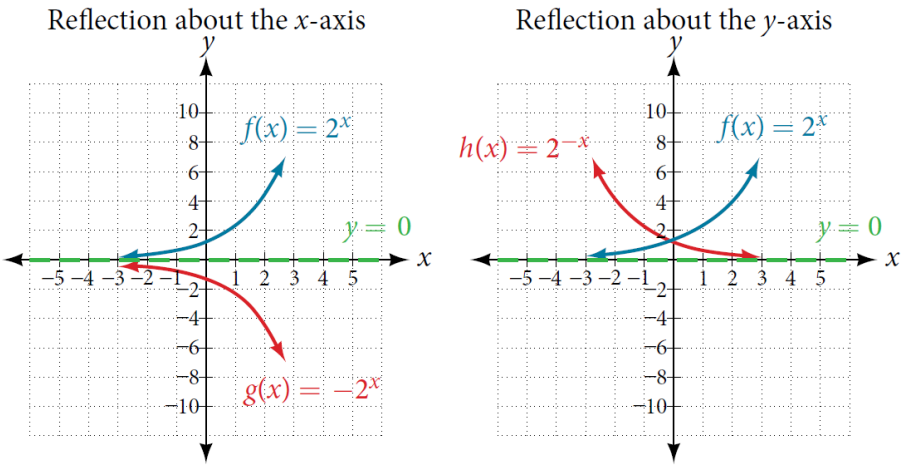
**Reflections of the parent function $f(x) = b^x$**

The function $f(x) = -b^x$

- reflects the parent function $f(x) = b^x$ about the $x$ -axis.
- has a $y$ -intercept of $(0, -1)$.
- has a range of $(-\infty, 0)$.
- has a horizontal asymptote at $y = 0$ and domain of $(-\infty, \infty)$, which are unchanged from the parent function.

The function $f(x) = b^{-x}$

- reflects the parent function $f(x) = b^x$ about the $y$ -axis.
- has a $y$ -intercept of $(0, 1)$, a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$, which are unchanged from the parent function.




(a) $g(x) = -2^x$ reflects the graph of $f(x) = 2^x$ about the $x$-axis. (b) $g(x) = 2^{-x}$ reflects the graph of $f(x) = 2^x$ about the $y$-axis.

## 11.1.2 Properties of Exponential Functions

Let $a$ and $b$ be positives, $a \neq 1, b \neq 1$, and $x$ and $y$ are real numbers:

- $a^x a^y = a^{x+y}$
- $\dfrac{a^x}{a^y} = a^{x-y}$
- $(ab)^x = a^x b^x$
- $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$
- $a^x = a^y$ if and only if $x = y$; $a \neq 1$.
- $a^x = b^x$ if and only if $a = b$; $x \neq 0$.

**Example 4**

Solve the following equations:

(a) $3^{x+1} = 9^{x-1}$

(b) $(3x)^2 = (2x+5)^2$

**Solution :**

(a) $3^{x+1} = 9^{x-1}$

$3^{x+1} = 3^{2(x-1)}$

$x + 1 = 2(x-1)$

$x + 1 = 2x - 2$

$-x = -3$

$x = 3$

(b) $(3x)^2 = (2x+5)^2$

$3x = 2x + 5$

$3x - 2x = 5$

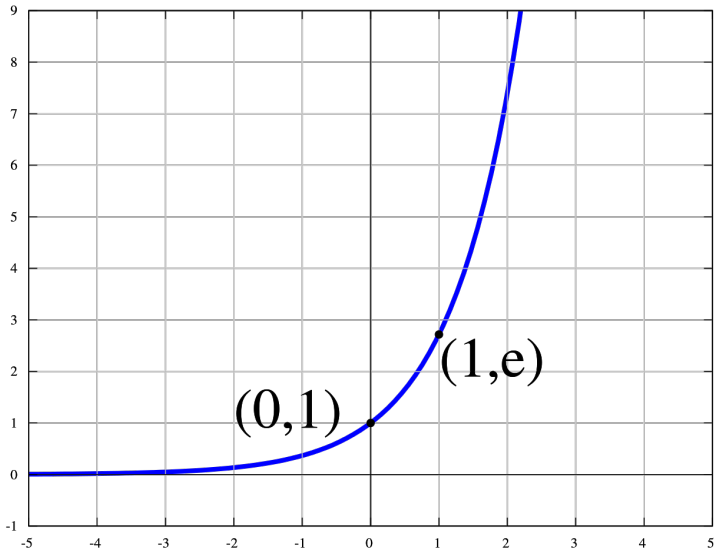
$x = 5$

## Natural Exponential Function

The exponential function is the function $f(x) = e^x$, where the base $e = 2.71828...$ is Euler's number and the argument $x$ occurs as an exponent.

The exponential function $f(x) = e^x$ is sometimes called the natural exponential function for distinguishing it from the other exponential functions.

The graph of the natural exponential function:




## Logarithmic Function

For $b > 1, b \neq 0$, the inverse of $f(x) = b^x$ denoted $f^{-1}(x) = \log_b x$ is the logarithmic function with base $b$.

$y = \log_b x$ (*logarithmic form*) is equivalent to $x = b^y$ (*exponential form*)

***How To* :: Convert logarithmic form to exponential form, vice versa**




Step 1. Examine the equation $y = \log_b(x)$ and identify $b, y$, and $x$.

Step 2. Rewrite $\log_b(x) = y$ as $b^y = x$. Identify the input values.

Step 3. To convert from exponents to logarithms, we follow the same steps in reverse. We identify the base $b$, exponent $x$, and output y. Then we write $x = \log_b(y)$.

**Example 5**

Write the following logarithmic equations in exponential form.

(a) $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$

(b) $\log_3(9) = 2$

**Solution :**

First, identify the values of $b, y$, and $x$. Then, write the equation in the form $b^y = x$.

(a) Here, $b = 6, y = \frac{1}{2}$, and $x = \sqrt{6}$.
Therefore, the equation $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$ is equivalent to $6^{\frac{1}{2}} = \sqrt{6}$

(b) Here, $b = 3, y = 2$, and $x = 9$.
Therefore, the equation $\log_3(9) = 2$ is equivalent to $3^2 = 9$.

**Example 6**

Write the following exponential equations in logarithmic form.

(a) $2^3 = 8$

(b) $5^2 = 25$

(c) $10^{-4} = \frac{1}{10000}$

**Solution :**

First, identify the values of $b, y$, and $x$. Then, write the equation in the form $y = \log_b x$.

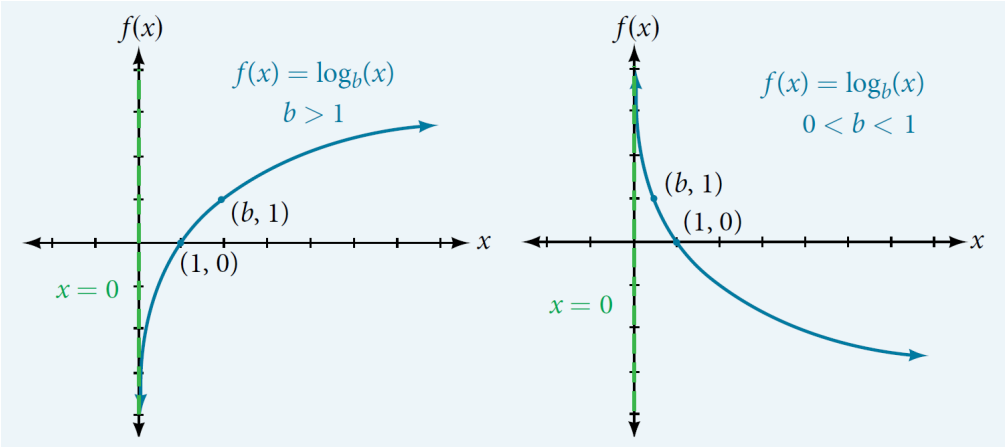
(a) Here, $b = 2, x = 3$, and $y = 8$.
Therefore, the equation $2^3 = 8$ is equivalent to $\log_2(8) = 3$

(b) Here, $b = 5, x = 2$, and $y = 25$.
Therefore, the equation $5^2 = 25$ is equivalent to $\log_5(25) = 2$

(c) Here, $b = 10, x = -4$, and $y = \frac{1}{10,000}$.
Therefore, the equation $10^{-4} = \frac{1}{10,000}$ is equivalent to $\log_{10}\left(\frac{1}{10000}\right) = -4$

### Characteristics of the graph of the parent function $f(x) = \log_b x$

For any real number x and constant $b > 0, b \neq 1$, we can see the following characteristics in the graph of $f(x) = \log_b(x)$ :




- one-to-one function
- vertical asymptote: $y = 0$
- domain: $(0, \infty)$
- range: $(-\infty, \infty)$
- $x$ -intercept: $(1, 0)$ and key point $(b, 1)$
- $y$ -intercept: none
- increasing if $b > 1$
- decreasing if $0 < b < 1$v

## 11.1.3 Graphing Logarithmic Functions

***How To* :: Graph the Logarithmic function of the form $f(x) = \log_b x$**

How?

Step 1. Draw and label the vertical asymptote, $x = 0$.

Step 2. Plot the $x$ -intercept, $(1, 0)$.

Step 3. Plot the key point $(b, 1)$.

Step 4. Draw a smooth curve through the points.

Step 5. State the domain, $(0, \infty)$, the range, $(-\infty, \infty)$, and the vertical asymptote, $x = 0$.

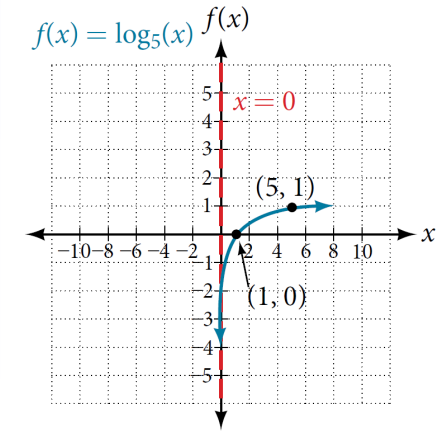
**Example 7**

Graph $f(x) = \log_5(x)$. State the domain, range, and asymptote.

**Solution :**

Before graphing, identify the behavior and key points for the graph.

- Since $b = 5$ is greater than one, we know the function is increasing. The left tail of the graph will approach the vertical asymptote $x = 0$, and the right tail will increase slowly without bound.
- The $x-$intercept is $(1, 0)$.
- The key point $(5, 1)$ is on the graph.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.

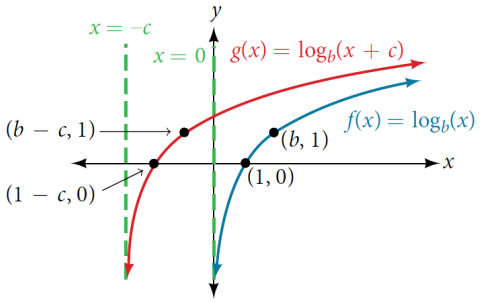
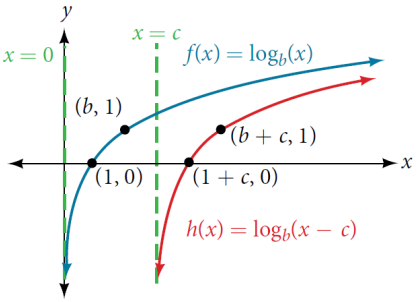



The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

**Horizontal shifts of the parent function $y = \log_b(x)$**

For any constant $c$, the function $f(x) = \log_b(x + c)$

- shifts the parent function $y = \log_b(x)$ left $c$ units if $c > 0$.
- shifts the parent function $y = \log_b(x)$ right $c$ units if $c < 0$.
- has the vertical asymptote $x = -c$.
- has domain $(-c, \infty)$.
- has range $(-\infty, \infty)$.

| **Shift left** $g(x) = \log_b(x + c)$ | **Shift right** $h(x) = \log_b(x - c)$ |
|---|---|
|  |  |
| • The asymptote changes to $x = -c$.<br>• The domain changes to $(-c, \infty)$.<br>• The range remains $(-\infty, \infty)$. | • The asymptote changes to $x = c$.<br>• The domain changes to $(c, \infty)$.<br>• The range remains $(-\infty, \infty)$. |



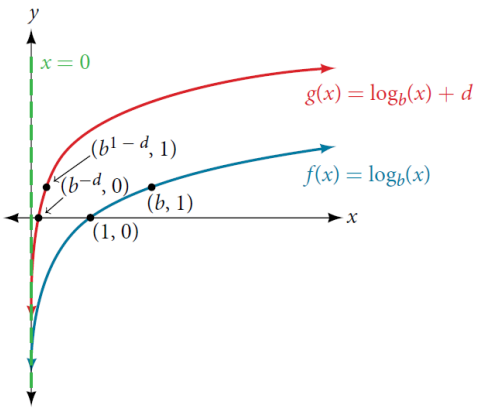
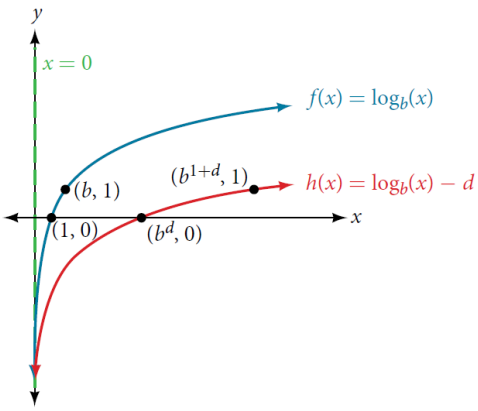


**Vertical shifts of the parent function $y = \log_b(x)$**

For any constant $d$, the function $f(x) = \log_b(x) + d$

- shifts the parent function $y = \log_b(x)$ up $d$ units if $d > 0$
- shifts the parent function $y = \log_b(x)$ down $d$ units if $d < 0$.

- has the vertical asymptote $x = 0$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

| **Shift up** $g(x) = \log_b(x) + d$ | **Shift down** $h(x) = \log_b(x) - d$ |
|---|---|
|  |  |
| • The asymptote remains $x = 0$.<br>• The domain remains to $(0, \infty)$.<br>• The range remains $(-\infty, \infty)$. | • The asymptote remains $x = 0$.<br>• The domain remains to $(0, \infty)$.<br>• The range remains $(-\infty, \infty)$. |





**Example 8**

Sketch a graph of $f(x) = \log_3(x) - 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

**Solution :**

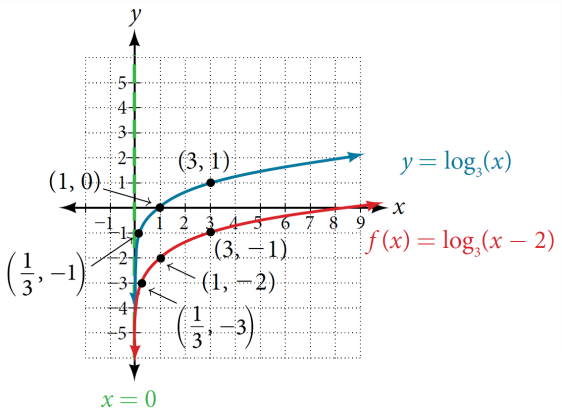
Since the function is $f(x) = \log_3(x) - 2$, we will notice $d = -2$. Thus $d < 0$. This means we will shift the function $f(x) = \log_3(x)$ down 2 units.

The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $\left(\frac{1}{3}, -1\right), (1, 0)$, and $(3, 1)$. The new coordinates are found by subtracting 2 from the $y$ coordinates.

Label the points $\left(\frac{1}{3}, -3\right), (1, -2)$, and $(3, -1)$.

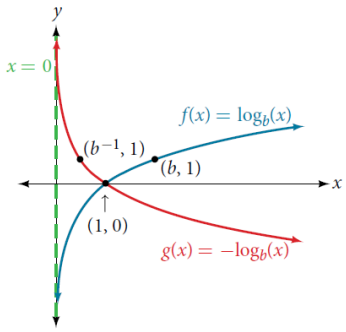
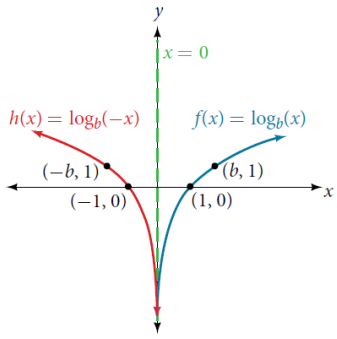
The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.



**Reflections of the parent function $y = \log_b(x)$**

The function $f(x) = -\log_b(x)$

- reflects the parent function $y = \log_b(x)$ about the $x$ -axis.
- has domain, $(0, \infty)$, range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function. The function $f(x) = \log_b(-x)$
- reflects the parent function $y = \log_b(x)$ about the $y$ -axis.
- has domain $(-\infty, 0)$.
- has range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

| **Reflection about the $x$-axis** $g(x) = \log_b(x),\ b > 1$ | **Reflection about the $y$-axis** $h(x) = \log_b(-x),\ b > 1$ |
|---|---|
|   |   |
| • The reflected function is decreasing as $x$ moves from zero to infinity.<br>• The asymptote remains $x = 0$.<br>• The $x$-intercept remains $(1, 0)$.<br>• The key point changes to $(b^{-1}, 1)$.<br>• The domain remains $(0, \infty)$.<br>• The range remains $(-\infty, \infty)$. | • The reflected function is decreasing as $x$ moves from infinity to zero.<br>• The asymptote remains $x = 0$.<br>• The $x$-intercept remains $(-1, 0)$.<br>• The key point changes to $(-b, 1)$.<br>• The domain changes to $(-\infty, 0)$.<br>• The range remains $(-\infty, \infty)$. |

**Example 9**

Sketch a graph of $f(x) = \log(-x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

**Solution :**

Before graphing $f(x) = \log(-x)$, identify the behavior and key points for the graph.

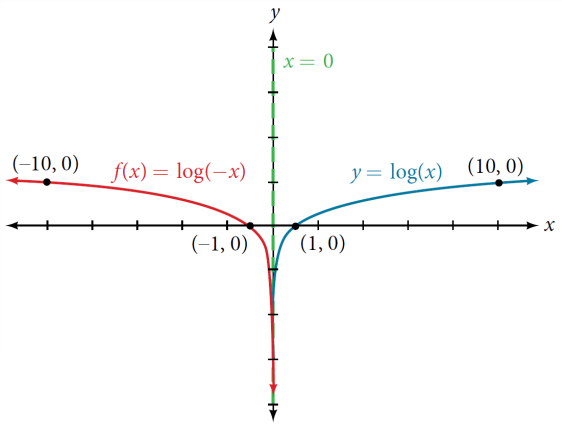
Since $b = 10$ is greater than one, we know that the parent function is increasing. Since the input value is multiplied by $-1$, $f$ is a reflection of the parent graph about the $y-$axis.

Thus, $f(x) = \log(-x)$ will be decreasing as $x$ moves from negative infinity to zero, and the right tail of the graph will approach the vertical asymptote $x = 0$.

The $x-$intercept is $(-1, 0)$.

We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.

The domain is $(-\infty, 0)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.




### 11.1.4 Properties of Logarithmic Functions

If $b, M, N$ are positive real numbers, $b \neq 0$, and $p$ and $x$ are real numbers then:

1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b b^x = x$
4. $b^{\log_b x} = x, x > 0$
5. $\log_b M = \log_b N$ if and only if $M = N$
6. $\log_b MN = \log_b M + \log_b N$
7. $\log_b \dfrac{M}{N} = \log_b M - \log_b N$
8. $\log_b M^p = p \log_b M$

- $\log_3 1 = 0$
- $\log_7 7 = 1$
- $\log_5 5^x = x$
- $4^{\log_4 x} = x, x > 0$
- $\log_3 2x = \log_3 x^2 \leftrightarrow 2x = x^2$
- $\log_6 10 = \log_6(2 \times 5) = \log_6 2 + \log_6 5$
- $\log_2 \dfrac{7}{2} = \log_2 7 - \log_2 2$
- $\log_6 6^3 = 3 \log_6 6 = 3$

**Note:**
$y = \log_{10} x = \log x$ (Common logarithmic function)
$y = \log_e x = \ln x$ (Natural logarithmic fumction)

**Example 10**

Evaluate the following using calculator

(a) $\log 2.335$ (b) $\log -3$ (c) $\ln 5.26$

**Solution :**

(a) $\log 2.335 = 0.3683$

(b) $\log -3 =$ Undefined because the domain of the logarithmic functions is the $(0, \infty)$.

(c) $\ln 5.26 = 1.660$

**Example 11**

Find the values of $x$ :

(a) $\log_5(x^2+1) = \log_5(-4x-3)$

(b) $\log_b(3x-1) - \log_b(x+2) = \log_b 2$

**Solution :**

(a) Given $\log_5(x^2+1) = \log_5(-4x-3)$

$$(x^2+1) = (-4x-3)$$
$$x^2+4x+4 = 0$$
$$(x+2)^2 = 0$$
$$x = -2, -2$$

(b) Given $\log_b(3x-1) - \log_b(x+2) = \log_b 2$

$$\log_b\left(\frac{3x-1}{x+2}\right) = \log_b 2$$
$$\therefore \frac{3x-1}{x+2} = 2$$
$$3x-1 = 2(x+2)$$
$$3x-1 = 2x+4$$
$$x = 5$$

## 11.1 EXERCISES

I. Sketch a graph of given exponential function. State the domain, range, and asymptote.

(1) $f(x) = 3\left(\frac{1}{2}\right)^x$ (2) $g(x) = -2(0.25)^x$ (3) $h(x) = 6(1.75)^{-x}$

(4) $f(x) = \frac{1}{2}(4)^x$ (5) $f(x) = 3(0.75)^x - 1$ (6) $f(x) = -4(2)^x + 2$

II. Sketch the graph of the logarithmic function. State the domain, range, and asymptote.

(1) $f(x) = \log_2(x+2)$ (2) $f(x) = 2\log(x)$ (3) $f(x) = \ln(-x)$

(4) $g(x) = \log(4x+16)+4$ (5) $g(x) = \log(6-3x)+1$ (6) $h(x) = -\frac{1}{2}\ln(x+1) - 3$

III. Solve for $x$:

(1) $3^{5x+1} = 27$ (2) $2^{3x+1} = \sqrt{2}$ (3) $8^{x+1} = 32\sqrt{2}$

(4) $4^{x-3} = 8$ (5) $27^{x+1} = 9$ (6) $7^{x^2} = 7^{2x+3}$

(7) $4^{5x-x^2} = 4^{-6}$ (8) $\left(\frac{1}{2}\right)^{x+4} = \frac{1}{2}^{3x-5}$ (9) $\left(\frac{1}{3}\right)^{2x-1} = \left(\frac{1}{3}\right)^{3-x}$

(10) $\left(\frac{4}{5}\right)^{6x+1} = \frac{5}{4}$ (11) $\left(\frac{1}{9}\right)^{5x+1} = 27$ (12) $\left(\frac{1}{8}\right)^{4x} = 16^{x+3}$

(13) $100^{2x+3} = 1000^{x+5}$ (14) $100^{x^2} = 1000^{10}$

IV. Rewrite the following in the equivalent exponential form.

(1) $\log_3 81 = 4$ (2) $\log_5 125 = 3$ (3) $\log_{10} 0.001 = -3$

(4) $\log_{10} 1000 = 3$ (5) $\log_6 \frac{1}{36} = -2$ (6) $\log_2 \frac{1}{64} = -6$

V. Rewrite the following in the equivalent logarithmic form.

(1) $8 = 4^{3/2}$ (2) $9 = 27^{2/3}$ (3) $\frac{1}{2} = 32^{-1/5}$

(4) $\frac{1}{8} = 2^{-3}$ (5) $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$ (6) $\left(\frac{5}{2}\right)^{-2} = 0.16$

VI. Simplify each expression.

(1) $\log_{16} 1$ (2) $\log_{25} 1$ (3) $\log_{0.5} 0.5$

(4) $\log_7 7$ (5) $\log_e e^4$ (6) $\log_{10} 10^5$

(7) $\log_4 256$ (8) $\log_{1/2} 2$ (9) $\log_{1/5}\left(\dfrac{1}{25}\right)$

(10) $e^{\log_e 5}$ (11) $e^{\log_e 10}$ (12) $\log_5 \sqrt[3]{5}$

(13) $\log_2 \sqrt{8}$

VII. Evaluate the following to four decimal places.

(1) $\log 49236$ (2) $\log 691450$ (3) $\ln 54.081$

(4) $\ln 19.722$ (5) $\log_7 13$ (6) $\log_9 78$

(7) $\log_5 120.24$ (8) $\log_{17} 304.66$

VIII. Find $x, y$, or $b$ as indicated.

(1) $\log_2 x = 2$ (2) $\log_3 x = 3$ (3) $\log_4 16 = y$

(4) $\log_8 64 = y$ (5) $\log_b 16 = 2$ (6) $\log_b 10^{-3} = -3$

(7) $\log_b 1 = 0$ (8) $\log_b b = 1$ (9) $\log_4 x = \dfrac{1}{2}$

(10) $\log_8 x = \dfrac{1}{3}$ (11) $\log_{1/3} 9 = y$ (12) $\log_{49}\left(\dfrac{1}{7}\right) = y$

(13) $\log_b 1000 = \dfrac{3}{2}$ (14) $\log_b 4 = \dfrac{2}{3}$ (15) $\log_8 x = -\dfrac{4}{3}$

(16) $\log_{25} x = -\dfrac{3}{2}$ (17) $\log_{16} 8 = y$ (18) $\log_9 27 = y$

IX. Rewrite the following expressions in terms of $\log x$ and $\log y$.

(1) $\log\left(\dfrac{x}{y}\right)$ (2) $\log(xy)$ (3) $\log\left(x^4y^3\right)$

(4) $\log\left(\dfrac{x^2}{\sqrt{y}}\right)$

X. Given that $\log x = -2$ and $\log y = 3$, find

(1) $\log\left(\dfrac{x}{y}\right)$ (2) $\log\left(\dfrac{\sqrt{x}}{y^3}\right)$ (3) $\log\left(x^5y^3\right)$

XI. Use the properties of logarithmic functions to write the expressions below as single logarithms.

(1) $\ln x - \ln y$ (2) $\log_3 x + \log_3 y$ (3) $2\ln x + 5\ln y - \ln z$

(4) $\dfrac{1}{2}\log_b(x+1) - 3\log_b(x-1)$ (5) $\log_5\left(x^2-1\right) - \log_5(x-1)$ (6) $2\left(\log_3 x + 2\log_3 y - 3\log_3 z\right)$

(7) $2\log_b(x-1) + \dfrac{1}{2}\log_b(x+1)$ (8) $\dfrac{1}{2}\log_b xy - 2\log_b(x+y)$

XII. Solve the following equations:

(1) $\log_5 x = 2$ (2) $\log(3x+5) = 2$ (3) $\log_2\left(x^2-x-2\right) = 2$

(4) $\log_5(x+1) - \log_5(x-1) = 2$ (5) $\log x + \log(x-3) = 1$ (6) $\ln(x-1) + \ln(x+2) = 1$

(7) $\log\left(x^2-2\right) = 3$ (8) $\log\left(x^2-2x\right) = 3$ (9) $\log_3 x + \log_3(x+2) = 1$

(10) $\log(2x+4) + \log(x-2) = 1$ (11) $\log(x-6) + \log(x+3) = 1$ (12) $\log(x+3) + \log x = 1$

(13) $\log(x-15) = 2 - \log x$ (14) $\log(x+1) = 2\log(x-1)$ (15) $\log_5 x + \log_5(x+1) = \log_5 20$

(16) $2\log x = \log 2 + \log(3x-4)$

# 11.2 Applications of Exponential and Logarithmic Functions

- **Population Doubling Time Model**

$$P = P_0 2^{\frac{t}{d}} \quad ;$$

where $P$ = population at time $t$, $P_0$ = population at time $t = 0$ and $d$ = doubling time.

- **Exponential Growth Model (given the relative growth rate)**

$$A = A_0 e^{kt} \quad ;$$

where $A$ = population at time $t$, $A_0$ = population at time $t = 0$ and $k$ = relative growth rate.

The relative growth rate is written as a percentage in decimal form.

- **Half - Life Decay Model**

$$A = A_0 2^{-\frac{t}{h}} \quad ;$$

where $A$ = amount at time $t$, $A_0$ = amount at time t = 0 and $h$ = Half-life.

- **Exponential Decay Model**

$$A = A_o e^{-kt} \quad ;$$

where $A$ = amount of radioactive material at time $t$, $A_0$ = amount at time $t = 0$ and $k$ = a positive constant specific to the type of material.

**Example 1**

The half-life of radium-226 is 1,590 years. How much of a 100 mg sample will be left in 500 years?

**Solution :**

If we start with 100 mg at the half-life there will be 50 mg remaining. We will use this information to find $k$.

Then we use that value of $k$ to help us find the amount of sample that will be left in 500 years.

$$\text{Given that } : A = 50, A_0 = 100, t = 1590 \text{ years}, k = ?$$

$$A = A_0 e^{kt}$$

Substitute the values in the formula and solve for $k$.

$$50 = 100 e^{1590k}$$

$$0.5 = e^{1590k}$$

Take the natural log of each side

$$\ln 0.5 = \ln e^{1590k}$$

$$\ln 0.5 = 1590k \ln e$$

$$\ln 0.5 = 1590k$$

$$\frac{\ln 0.5}{1590} = k$$

We use this rate of growth to predict the amount that will be left in 500 years.

$$t = 500 \text{ years }, A = ?$$

$$A = A_0 e^{kt} \text{ then } A = 100 e^{\left(\frac{\ln 0.5}{1590}\right)500}$$

$$\therefore A \approx 80.4 \text{ mg}$$

In 500 years there would be approximately 80.4 mg remaining.

**Example 2**

Researchers recorded that a certain bacteria population grew from 100 to 300 in 3 hours. At this rate of growth, how many bacteria will there be 24 hours from the start of the experiment?

**Solution :**

We require two steps.

- First we must find the unknown rate, $k$.
- Use the value of $k$ to find the unknown number of bacteria.

Given that: $A = 300, A_0 = 100, t = 3$ hours, $k =?$

$$A = A_0 e^{kt}$$
$$300 = 100e^{3k}$$
$$3 = e^{3k}$$

Take the natural log of each side

$$\ln 3 = \ln e^{3k}$$
$$\ln 3 = 3k \ln e$$
$$\ln 3 = 3k$$
$$\frac{\ln 3}{3} = k$$
$$k \approx 0.366$$

We use this rate of growth to predict the number of bacteria there will be in 24 hours.

$$t = 24 \text{ hours }, A =?$$
$$A = A_0 e^{kt} \text{ then } A = 100e^{(\frac{\ln 3}{8})24}$$
$$\therefore A \approx 656100$$

At this rate of growth, they can expect $656,100$ bacteria.

**Example 3**

According to a 2008 estimate, the population of Nicaragua was about 5.7 million, and that population is growing due to a high birth rate and relatively low mortality rate. If the population continues to grow at the current rate, it will double in 37 years. If the growth remains steady, what will the population be in 15 years? Calculate answers to three significant digits.

**Solution :**

We can use the doubling time growth model, $P = P_0 2^{\frac{t}{d}}$

Given that: $P_0 = 5.7$ million, $d = 37$, $t = 15$ years

$$P = 5.7(2)^{\frac{15}{37}}$$
$$P \approx 7.55 \text{ million}$$

**Example 4**

The radioactive isotope gallium 67 (67Ga), used in the diagnosis of malignant tumors, has a biological half-life of 46.5 hours. If we start with 100 milligrams of the isotope, how many milligrams will be left after 24 hours?

**Solution :**

Given that: $A_0 = 100, \quad t = 24, \quad h = 46.5$

$$A = A_0 2^{-\frac{t}{h}}$$

$$A = 100 \times 2^{-\frac{24}{46.5}}$$

$$= 69.9 \text{ Milligrams}$$

## 11.2 EXERCISES

I. Answer the following questions:

(1) Write an exponential equation describing the given population at any time $t$.
a) Initial population 200; doubling time 5 months
b) Initial population 5,000; doubling time 3 years
c) Initial population 2,000; continuous growth at 2% per year
d) Initial population 500; continuous growth at 3% per week

(2) Write an exponential equation describing the amount of radioactive material present at any time.
a) Initial amount 100 grams; half-life 6 hours
b) Initial amount 5 pounds; half-life 1,300 years
c) Initial amount 4 kilograms ; continuous decay at 12.4% per year
d) Initial amount 50 milligrams ; continuous decay 0.03% per year

(3) According to a 2008 estimate, the population of Nicaragua was about 5.7 million, and that population is growing due to high birth rate and relatively low mortality rate. If the population continues to grow at the current rate, it will double in 37 years. If the growth remains steady, what will the population be in (A) 15 years ? (B) 40 years ?

(4) The use of insecticide DDT is no longer allowed in many countries because of its long-term adverse effects.If a farmer uses 25 pounds of active DDT, assuming half-life is 12 years, how much will still be active after (A) 5 years? (B) 20 years ?

(5) The half-life of Plutonium-241 is 13 years. How much of an initial 2g sample will remain after 5 years?

(6) At the start of an experiment, 2000 bacteria are present in a colony. Two hours later, the population is 3800.
a) Determine the growth constant $k$.
b) Determine the population five hours after the start of the experiment.
c) When will the population reach $10,000$?

(7) There are $2 \times 10^4$ bacteria present in a colony at the beginning of an experiment. Eight hours later, the population is $3 \times 10^4$
a) Determine the growth constant $k$.
b) What was the population two hours after the start of the experiment?
c) How long will it take for the population to triple?

(8) The half-life of Sodium-24 is 14.9 hours. Suppose that a hospital buys a 40g sample of Sodium-24.
a) How much of the sample will remain after 48 hours?
b) How long will it be until only one gram remains?

(9) Suppose \$2000 is invested in an account which offers 7.125% compounded monthly.
a) Express the amount $A$ in the account as a function of the term of the investment $t$ in years.
b) How much is in the account after 5 years?
c) How long will it take for the initial investment to double?

# Chapter 12

# Trigonometric Functions

## Contents


**12.1 Angles and Circle**

12.1.1 Angles

12.1.2 Converting Between Radians and Degrees

12.1.3 Determining the Length of an Arc and Area of a Sector

**12.2 Trigonometric Ratios and Identities**

12.2.1 Using Right Triangles to Evaluate Trigonometric Ratios

12.2.2 Fundamental Trigonometric Identities

**12.3 Applications of Trigonometry**

**12.4 Circular Functions**

**12.5 Sum and Difference Identities**


**Learning outcomes covered:**

**h.** Know the relationship between degree and radian measure of an angle and find the length of a circular arc and the area of a sector.

**i.** Understand trigonometric and circular functions and use the fundamental trigonometric identities in various problems.

**j.** Solve a right angle triangle using angle of elevation and depression.

**k.** Apply knowledge of basic algebra and trigonometry in real life problems.

**u.** Understand the definition of the different types of angles and measure them in degrees and radians.

**v.** Describe analytically the trigonometric and circular functions.

**r.** Demonstrate an understanding of trigonometric identities.

**Learning Objectives:**

**By the end of this chapter, the students will be able to:**

- Know the relationship between degree and radian measure of an angle

- Find the length of a circular arc and the area of a sector
- Understand trigonometric and circular functions
- Use the fundamental trigonometric identities
- Understand angle of elevation and depression
- Apply knowledge of trigonometry in real life problems
- Understand the different types of angles.
- Measure the angles in degree and radians.
- Understand basic trigonometric functions and their properties.
- Verify the fundamental trigonometric identities.

## Introduction

A golfer swings to hit a ball over a sand trap and onto the green. An airline pilot maneuvers a plane toward a narrow runway. A dress designer creates the latest fashion. What do they all have in common? They all work with angles, and so do all of us at one time or another. Sometimes we need to measure angles exactly with instruments. Other times we estimate them or judge them by eye. Either way, the proper angle can make the difference between success and failure in many undertakings. In this section, we will examine properties of angles.

# 12.1 Angles and Circle

### 12.1.1 Angles

A golfer swings to hit a ball over a sand trap and onto the green. An airline pilot maneuvers a plane toward a narrow runway. A dress designer creates the latest fashion. What do they all have in common? They all work with angles, and so do all of us at one time or another. Sometimes we need to measure angles exactly with instruments. Other times we estimate them or judge them by eye. Either way, the proper angle can make the difference between success and failure in many undertakings. In this section, we will examine properties of angles.

#### Drawing Angles in Standard Position

Properly defining an angle first requires that we define a ray. A **ray** consists of one point on a line and all points extending in one direction from that point. The first point is called the **endpoint** of the ray. We can refer to a specific ray by stating its endpoint and any other point on it. The ray in Figure 1 can be named as ray EF, or in symbol form $\overrightarrow{EF}$.

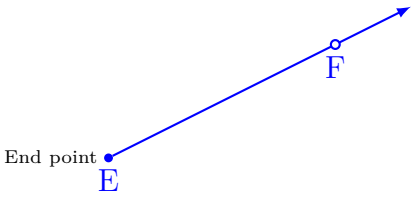



**Figure 1:** Ray EF

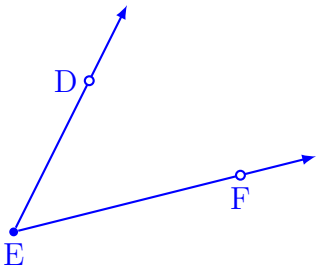



**Figure 2**: Angle DEF

An angle is the union of two rays having a common endpoint. The endpoint is called the vertex of the angle, and the two rays are the sides of the angle. The angle in Figure 2 is formed from $\overrightarrow{ED}$ and $\overrightarrow{EF}$. Angles can be named using a point on each ray and the vertex, such as angle DEF, or in symbol form $\angle DEF$. Greek letters are often used as variables for the measure of an angle. Table 1 is a list of Greek letters commonly used to represent angles, and a sample angle is shown in Figure 3.

| $\theta$ | $\phi$ or $\varphi$ | $\alpha$ | $\beta$ | $\gamma$ |
|---|---|---|---|---|
| theta | phi | alpha | beta | gamma |

Table 1

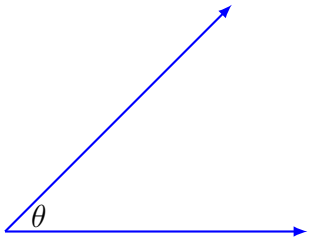



**Figure 3**: Angle theta, shown as $\angle\theta$

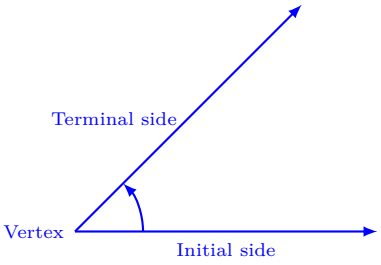



**Figure 4**

Angle creation is a dynamic process. We start with two rays lying on top of one another. We leave one fixed in place, and rotate the other. The fixed ray is the **initial** side, and the rotated ray is the **terminal** side. In order to identify the different sides, we indicate the rotation with a small arc and arrow close to the vertex as in Figure 4.

### Measure of Angles

### 1. Degree

The **measure of an angle** is the amount of rotation from the initial side to the terminal side. Probably the most familiar unit of angle measurement is the degree. One **degree** is $\frac{1}{360}$ of a circular rotation, so a complete circular rotation contains **360 degrees**. An angle measured in degrees should always include the unit degrees after the number, or include the degree symbol °. For example, 90 degrees = 90°.

If the angle is measured in a counterclockwise direction from the initial side to the terminal side, the angle is said to be a **positive angle**. If the angle is measured in a clockwise direction, the angle is said to be a **negative angle**.

**2. Radian**

One radian is the measure of the central angle of a circle such that the length of the arc between the initial side and the terminal side is equal to the radius of the circle. A full revolution (360°) equals $2\pi$ radians. A half revolution (180°) is equivalent to $\pi$ radians.

*The radian measure of an angle is the ratio of the length of the arc subtended by the angle to the radius of the circle.*

In other words, if $s$ is the length of an arc of a circle, and $r$ is the radius of the circle, then the central angle containing that arc measures $\frac{s}{r}$ radians. In a circle of radius 1, the radian measure corresponds to the length of the arc.

Note that when an angle is described without a specific unit, it refers to radian measure.

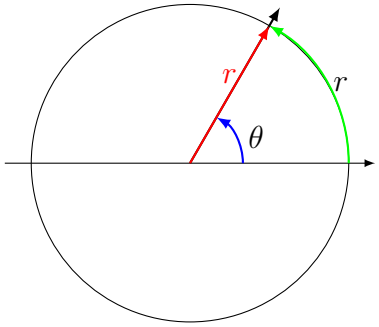



**Figure 5: The angle $\theta$ sweeps out a measure of one radian. Note that the length of the intercepted arc is the same as the length of the radius of the circle.**

### 12.1.2 Converting Between Radians and Degrees

Because degrees and radians both measure angles, we need to be able to convert between them. We can easily do so using a proportion where $\theta$ is the measure of the angle in degrees and $\theta_R$ is the measure of the angle in radians.

$$\frac{\theta}{180} = \frac{\theta_R}{\pi}$$

This proportion shows that the measure of angle $\theta$ in degrees divided by 180 equals the measure of angle $\theta$ in radians divided by $\pi$. Or, phrased another way, degrees is to 180 as radians is to $\pi$.

$$\frac{\text{Degrees}}{180} = \frac{\text{Radians}}{\pi}$$

**Example 1:** *Convert each radian measure to degrees.*

***a.*** $\frac{\pi}{6}$ ***b.*** $\frac{2\pi}{5}$

**Solution:** Because we are given radians and we want degrees, we take $\frac{\text{Degrees}}{180} = \frac{\text{Radians}}{\pi}$.

**a.** $\theta_R = \dfrac{\pi}{6}$.

$$\frac{\theta}{180} = \frac{\theta_R}{\pi} \Rightarrow \frac{\theta}{180} = \frac{\frac{\pi}{6}}{\pi}$$

$$\Rightarrow \frac{\theta}{180} = \frac{\cancel{\pi}1}{6\cancel{\pi}}$$

$$\Rightarrow \theta = \frac{180}{6} = 30^\circ$$

**b.** $\theta_R = \dfrac{2\pi}{5}$

$$\frac{\theta}{180} = \frac{\theta_R}{\pi} \Rightarrow \frac{\theta}{180} = \frac{\frac{2\pi}{5}}{\pi}$$

$$\Rightarrow \frac{\theta}{180} = \frac{2\cancel{\pi}1}{5\cancel{\pi}}$$

$$\Rightarrow \theta = \frac{2 \times 180}{5} = 72^\circ$$

**Example 2:** *Convert each degree measure to radians:* ***a.*** *15 degrees* ***b.*** *420°*

**Solution:** Because we want to convert degree to radian, we take $\dfrac{\text{Radians}}{\pi} = \dfrac{\text{Degrees}}{180}$.

**a.** $\theta = 15$ degrees

$$\frac{\theta_R}{\pi} = \frac{\theta}{180} \Rightarrow \frac{\theta_R}{\pi} = \frac{15}{180}$$

$$\Rightarrow \theta_R = \frac{15\pi}{180}$$

$$\Rightarrow \theta = \frac{\pi}{12}$$

**b.** $\theta = 420^\circ$

$$\frac{\theta_R}{\pi} = \frac{\theta}{180} \Rightarrow \frac{\theta_R}{\pi} = \frac{420}{180}$$

$$\Rightarrow \theta_R = \frac{420\pi}{180}$$

$$\Rightarrow \theta = \frac{7\pi}{3}$$

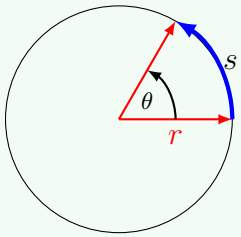
### 12.1.3 Determining the Length of an Arc and Area of a Sector

#### Determining the Length of an Arc

***Arc length on a circle***

In a circle of radius $r$, the length of an arc $s$ subtended by an angle with measure $\theta$ in radians is

$$s = r\theta$$




*How To*

Given a circle of radius $r$, calculate the length $s$ of the arc subtended by a given angle of measure $\theta$.

1. If necessary, convert $\theta$ to radians.
2. Multiply the radius $r$ by the radian measure of $\theta$ : $s = r\theta$.

**Example 3:** *Find the length of the arc of a circle of radius 12 inches subtended by a central angle of* $\dfrac{\pi}{4}$ *radians.*

**Solution:**

$$\text{Central angle, } \theta = \frac{\pi}{4}\text{radians}$$

$$\text{radius, } r = 12 \text{ inches}$$

$$\text{Arc length, } s = r\theta$$

$$= 12 \cdot \frac{\pi}{4} = 3\pi \text{ inches}$$

**Example 4:** *Find the arc length along a circle of radius 10 units subtended by an angle of* $120°$.

**Solution:** First, we need to convert the angle measure into radians.

$$120° = \frac{120 \cdot \pi}{180} = \frac{2\pi}{3} \text{ radians}$$

$$\text{Central angle, } \theta = \frac{2\pi}{3} \text{radians}$$

$$\text{radius, } r = 10 \text{ units}$$

$$\text{Arc length, } s = r\theta$$

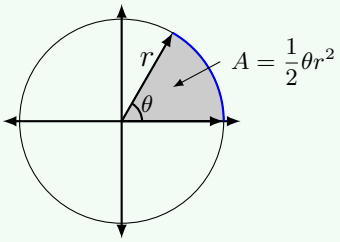
$$= 10 \cdot \frac{2\pi}{3} = \frac{20\pi}{3} \text{ units}$$

## Finding the Area of a Sector of a Circle

***Area of a sector***

The **area of a sector** of a circle with radius r subtended by an angle $\theta$, measured in **radians**, is

$$A = \frac{1}{2}\theta r^2$$

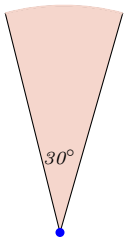



***How To . . .***

Given a circle of radius $r$, find the area of a sector defined by a given angle $\theta$.

1. If necessary, convert $\theta$ to radians.
2. Multiply half the radian measure of $\theta$ by the square of the radius $r$: $A = \frac{1}{2}\theta r^2$.

**Example 5:** *An automatic lawn sprinkler sprays a distance of 20 feet while rotating 30 degrees, as shown in the figure. What is the area of the sector of grass the sprinkler waters?*




**Solution:** First, we need to convert the angle measure into radians.

$$30 \text{ degrees} = 30 \cdot \frac{\pi}{180} = \frac{\pi}{6}\text{radians}$$

The area of the sector is then $Area = \frac{1}{2}\theta r^2$

$$= \frac{1}{2}\left(\frac{\pi}{6}\right)(20)^2$$

$$= \frac{100\pi}{3}\,\text{ft}^2$$

## 12.1 EXERCISES

*For the following exercises, convert angles in radians to degrees.*

**1.** $\frac{3\pi}{4}$ radians **2.** $\frac{\pi}{9}$ radians **3.** $-\frac{5\pi}{12}$ radians

**4.** $\frac{\pi}{3}$ radians

*For the following exercises, convert angles in degrees to radians.*

**5.** $90°$ **6.** $210°$ **7.** $-540°$

**8.** $-120°$

*For the following exercises, use to given information to find the length of a circular arc. Round to two decimal places.*

**9.** Find the length of the arc of a circle of radius 9 miles subtended by the central angle of $\frac{\pi}{3}$.

**10.** Find the length of the arc of a circle of diameter 15 meters subtended by the central angle of $\frac{11\pi}{6}$.

**11.** Find the length of the arc of a circle of radius 10 centimeters subtended by the central angle of $50°$.

**12.** Find the length of the arc of a circle of radius 5 inches subtended by the central angle of $220°$.

*For the following exercises, use the given information to find the* ***area*** *of the sector. Round to four decimal places.*

**13.** A sector of a circle has a central angle of $45°$ and a radius 6 cm.

**14.** A sector of a circle has a central angle of $30°$ and a radius 20 cm.

**15.** A sector of a circle with diameter 10 feet and an angle of $\frac{\pi}{2}$ radians.

**16.** A sector of a circle with radius of 0.7 inches and an angle of $\pi$ radians.

# 12.2 Trigonometric Ratios and Identities

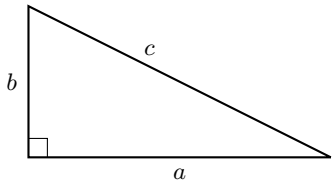
In this section, we will define a new group of functions known as trigonometric functions In this section, we will begin an examination of the fundamental trigonometric identities, including how we can verify them and how we can use them to simplify trigonometric expressions.

## 12.2.1 Using Right Triangles to Evaluate Trigonometric Ratios

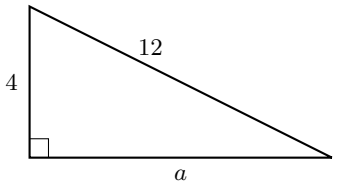
One of the most famous formulas in mathematics is the Pythagorean Theorem. It is based on a right triangle, and states the relationship among the lengths of the sides. It has immeasurable uses in architecture, engineering, the sciences, geometry, trigonometry, and algebra, and in everyday applications.

### Pythagorean Theorem

The Pythagorean Theorem is given as $a^2 + b^2 = c^2$ where $a$ and $b$ refer to the legs of a right triangle adjacent to the 90° angle, and $c$ refers to the hypotenuse, as shown in the figure.




**Example 6:** *Find the length of the missing side of the right triangle in the figure given below.*




**Solution:** As we have measurements for side $b$ and the hypotenuse, the missing side is $a$.

$$a^2 + b^2 = c^2$$

$$a^2 + 4^2 = 12^2$$

$$a^2 + 16 = 144$$

$$a^2 = 128$$

$$a = \sqrt{128} = \sqrt{2 \times 64}$$

$$= 8\sqrt{2}$$

## Trigonometric Ratios

We can define the trigonometric ratios in terms an angle $\theta$ and the lengths of the sides of the triangle. The adjacent side is the side closest to the angle, $\theta$. (Adjacent means next to.) The opposite side is the side across from the angle, $\theta$. The hypotenuse is the side of the triangle opposite the right angle. These sides are labeled in Figure 1.

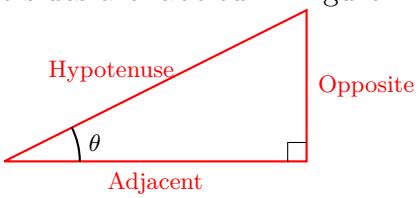



**Figure 1: The sides of a right triangle in relation to angle $\theta$.**

Given a right triangle with an acute angle of $\theta$, the first three trigonometric ratios are listed.

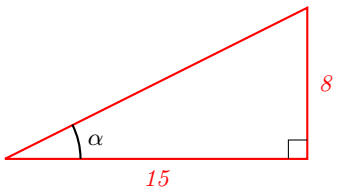
| **Sine** | **Cosine** | **Tangent** |
|---|---|---|
| $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ | $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ | $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$ |

*How To*

Given the side lengths of a right triangle and one of the acute angles, find the sine, cosine, and tangent of that angle.

1. Find the sine as the ratio of the opposite side to the hypotenuse.
2. Find the cosine as the ratio of the adjacent side to the hypotenuse.
3. Find the tangent as the ratio of the opposite side to the adjacent side.

**Example 7:** *Given the triangle shown in the figure, find the values of all trigonometric ratios of the angle $\alpha$.*




**Solution:** The side adjacent to the angle is 15, and opposite to the angle is 8. Let $c$ be the hypotenuse. By the Pythagorean Theorem,

$$c^2 = 15^2 + 8^2 = 225 + 64 = 289$$

$$c = \sqrt{289} = 17$$

$$\sin\alpha = \frac{\text{opp}}{\text{hyp}} = \frac{8}{17} \qquad \csc\alpha = \frac{17}{8}$$

$$\cos\alpha = \frac{\text{adj}}{\text{hyp}} = \frac{15}{17} \qquad \sec\alpha = \frac{17}{15}$$

$$\tan\alpha = \frac{\text{opp}}{\text{adj}} = \frac{8}{15} \qquad \cot\alpha = \frac{15}{8}$$

## Reciprocal Functions

| Secant | Cosecant | Cotangent |
|---|---|---|
| $\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent}}$ | $\csc\theta = \dfrac{\text{hypotenuse}}{\text{opposite}}$ | $\cot\theta = \dfrac{\text{adjacent}}{\text{opposite}}$ |

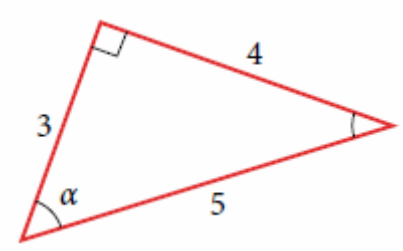
Take another look at these definitions. These functions are the reciprocals of the first three functions.

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

**Example 8:** *Using the triangle shown in Figure, evaluate* $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, $\csc\alpha$, *and* $\cot\alpha$.




**Solution:**

$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5} \qquad \csc\alpha = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{4}$$

$$\cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5} \qquad \sec\alpha = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{3}$$

$$\tan\alpha = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3} \qquad \cot\alpha = \frac{\text{adjacent}}{\text{opposite}} = \frac{3}{4}$$

Another approach would have been to find sine, cosine, and tangent first. Then find their reciprocals to determine the other functions.

$$\csc\alpha = \frac{1}{\sin\alpha} = \frac{1}{\frac{4}{5}} = \frac{5}{4}$$

$$\sec\alpha = \frac{1}{\cos\alpha} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$$

$$\cot\alpha = \frac{1}{\tan\alpha} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

### 12.2.2 Fundamental Trigonometric Identities

Identities enable us to simplify complicated expressions. They are the basic tools of trigonometry used in solving trigonometric equations. In fact, we use algebraic techniques constantly to simplify trigonometric expressions. We already know that all of the trigonometric functions are related because they all are defined in terms of the unit circle. Consequently, any trigonometric identity can be written in many ways.

In this section, we will work with the fundamental identities: the Pythagorean identities, the even-odd identities, the reciprocal identities, and the quotient identities.

| **Pythagorean Identities** | | |
|---|---|---|
| $\sin^2\theta + \cos^2\theta = 1$ | $1 + \cot^2\theta = \csc^2\theta$ | $1 + \tan^2\theta = \sec^2\theta$ |

**Table 1**

The next set of fundamental identities is the set of **even-odd identities**. The even-odd identities relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle. (See Table 2).

| **Even Identity** | **Odd Identities** | |
|---|---|---|
| $\cos(-\theta) = \cos\theta$ | $\sin(-\theta) = -\sin\theta$ | $\tan(-\theta) = -\tan\theta$ |
| $\sec(-\theta) = \sec\theta$ | $\csc(-\theta) = -\csc\theta$ | $\cot(-\theta) = -\cot\theta$ |

**Table 2**

The next set of fundamental identities is the set of **reciprocal identities**, which, as their name implies, relate trigonometric functions that are reciprocals of each other. See Table 3.

| **Reciprocal Identities** | |
|---|---|
| $\sin\theta = \dfrac{1}{\csc\theta}$ | $\csc\theta = \dfrac{1}{\sin\theta}$ |
| $\cos\theta = \dfrac{1}{\sec\theta}$ | $\sec\theta = \dfrac{1}{\cos\theta}$ |
| $\tan\theta = \dfrac{1}{\cot\theta}$ | $\cot\theta = \dfrac{1}{\tan\theta}$ |

**Table 3**

The final set of identities is the set of **quotient identities**, which define relationships among certain trigonometric functions and can be very helpful in verifying other identities. See Table 4.

| Quotient Identities | |
|---|---|
| $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ | $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ |

**Table 4**

**Example 9:** *Verify* $\tan\theta\cos\theta = \sin\theta$.

**Solution:** We will start on the left side, as it is the more complicated side:

$$\begin{aligned}\tan\theta\cos\theta &= \left(\frac{\sin\theta}{\cos\theta}\right)\cos\theta \\ &= \left(\frac{\sin\theta}{\cancel{\cos\theta}}\right)\cancel{\cos\theta} \\ &= \sin\theta\end{aligned}$$

**Example 10:** *Verify the following equivalency using the even-odd identities:*

$(1+\sin x)[1+\sin(-x)] = \cos^2 x$

**Solution:** Working on the left side of the equation, we have

$$\begin{aligned}(1+\sin x)[1+\sin(-x)] &= (1+\sin x)(1-\sin x) && \text{Since } \sin(-x) = -\sin x \\ &= 1-\sin^2 x && \text{Difference of squares} \\ &= \cos^2 x && \left[\cos^2 x = 1-\sin^2 x\right]\end{aligned}$$

**Example 11:** *Verify the identity* $\dfrac{\sec^2\theta - 1}{\sec^2} = \sin^2\theta$.

**Solution:** As the left side is more complicated, lets begin there.

$$\begin{aligned}
\frac{\sec^2\theta - 1}{\sec^2\theta} &= \frac{(\tan^2\theta + 1) - 1}{\sec^2\theta} && \left[\sec^2\theta = \tan^2\theta + 1\right] \\
&= \frac{\tan^2\theta}{\sec^2\theta} \\
&= \tan^2\theta\left(\frac{1}{\sec^2\theta}\right) \\
&= \tan^2\theta(\cos^2\theta) && \left[\cos^2\theta = \frac{1}{\sec^2\theta}\right] \\
&= \frac{\sin^2\theta}{\cos^2\theta}(\cos^2\theta) && \left[\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}\right] \\
&= \frac{\sin^2\theta}{\cancel{\cos^2\theta}}(\cancel{\cos^2\theta}) \\
&= \sin^2\theta
\end{aligned}$$

---

**Example 12:** *Verify the identity* $(1-\cos^2 x)(1+\cot^2 x) = 1$.

**Solution:** We will work on the left side of the equation

$$\begin{aligned}
(1-\cos^2 x)(1+\cot^2 x) &= (1-\cos^2 x)\left(1 + \frac{\cos^2 x}{\sin^2 x}\right) \\
&= (1-\cos^2 x)\left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x}\right) \\
&= (\sin^2 x)\left(\frac{1}{\sin^2 x}\right) \\
&= 1
\end{aligned}$$

---

## 12.2 EXERCISES

### Graphical

**1.** *For the following exercises, use Figure 1 to evaluate each trigonometric ratio of angle $\theta$.*

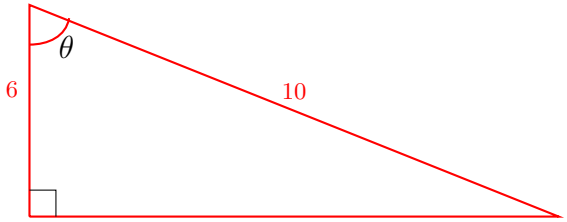



Figure 1

**a.** $\sin\theta$ **d.** $\csc\theta$
**b.** $\cos\theta$ **e.** $\sec\theta$
**c.** $\tan\theta$ **f.** $\cot\theta$

**2.** *For the following exercises, use Figure 2 to evaluate each trigonometric ratio of angle $\alpha$.*

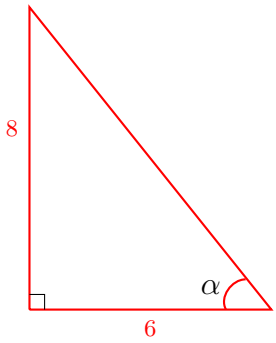



Figure 2

**a.** $\sin\alpha$ **d.** $\csc\alpha$
**b.** $\cos\alpha$ **e.** $\sec\alpha$
**c.** $\tan\alpha$ **f.** $\cot\alpha$

### Verifying Trigonometric Identities

*For the following exercises, use the fundamental identities to fully simplify the expression.*

**3.** $\sin x\cos x\sec x$

**4.** $\sin(-x)\cos(-x)\csc(-x)$

**5.** $\tan x\sin x+\sec x\cos^2 x$

**6.** $\csc x+\cos x\cot(-x)$

**7.** $\dfrac{\cot t+\tan t}{\sec(-t)}$

**8.** $3\sin^3 t\csc t+\cos^2 t+2\cos(-t)\cos t$

**9.** $-\tan(-x)\cot(-x)$

**10.** $\dfrac{-\sin(-x)\cos x\sec x\csc x\tan x}{\cot x}$

**11.** $\dfrac{1-\cos^2 x}{\tan^2 x}+2\sin^2 x$

*For the following exercises, verify the identity.*

**12.** $\cos x-\cos^3 x=\cos x\sin^2 x$

**13.** $\cos x(\tan x-\sec(-x))=\sin x-1$

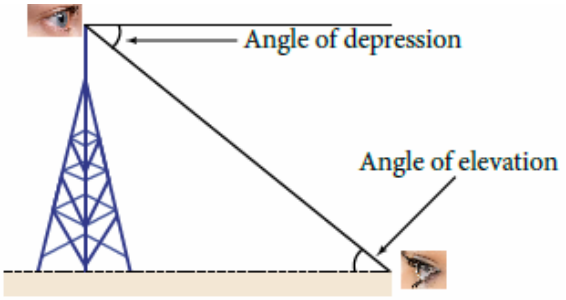
**14.** $\dfrac{1+\sin^2 x}{\cos^2 x}=1+2\tan^2 x$

**15.** $(\sin x+\cos x)^2=1+2\sin x\cos x$

**16.** $\cos^2 x-\tan^2 x=2-\sin^2 x-\sec^2 x$

## 12.3 Applications of Trigonometry

Right-triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height.




The **angle of elevation** of an object above an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye.

The **angle of depression** of an object below an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye.

1. Make a sketch of the problem situation to keep track of known and unknown information.
2. Lay out a measured distance from the base of the object to a point where the top of the object is clearly visible.
3. At the other end of the measured distance, look up to the top of the object. Measure the angle the line of sight makes with the horizontal.
4. Write an equation relating the unknown height, the measured distance, and the tangent of the angle of the line of sight.
5. Solve the equation for the unknown height.

**Example 13:** *To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an angle of* $60°$ *between a line of sight to the top of the tree and the ground, as shown in Figure. Find the height of the tree.*

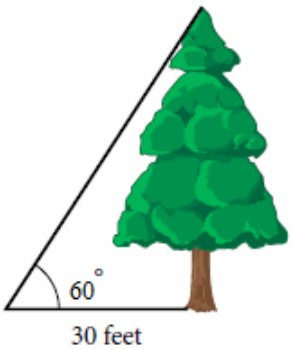
$$\left[\sin 60° = \frac{\sqrt{3}}{2},\ \cos 60° = \frac{1}{2},\ \tan 60° = \sqrt{3}\right]$$

**Solution:**

We have that the angle of elevation is $60°$ and the adjacent side is 30 ft long. The opposite side is the unknown height. Let the height $= h$.

The trigonometric function relating the side opposite to an angle and the side adjacent to the angle is the tangent.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$



$$\tan(60^\circ) = \frac{h}{30}$$

$$h = 30\tan(60^\circ)$$

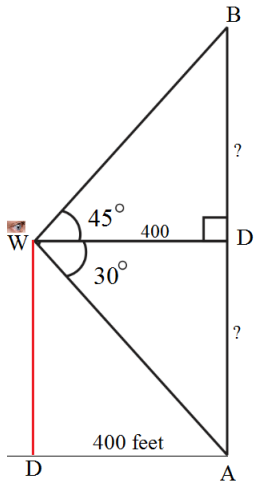
$$h = 30\sqrt{3}$$

The tree is $30\sqrt{3}$ feet tall.

**Example 14:** *A radio tower is located 400 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is* $45^\circ$*, and that the angle of depression to the bottom of the tower is* $30^\circ$*. How tall is the tower?*

**Solution:**

Let $AB$ be the radio tower, where
$A$ is the base and $B$ is the top.
Let $W$ be the window in the building.
Draw a line $WD$ perpendicular to $AB$.




$$AB = AD + DB$$

We have,

$$AD = WD = 400$$

angle of elevation, $\angle DWB = 45^\circ$ and
angle of depression, $\angle DWA = 30^\circ$.

**From the triangle, BWD,**

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan(45^\circ) = \frac{BD}{400}$$

$$1 = \frac{BD}{400}$$

$$BD = 400 \text{ feet}$$

**From the triangle, ADW,**

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan(30^\circ) = \frac{AD}{400}$$

$$\frac{1}{\sqrt{3}} = \frac{AD}{400}$$

$$AD = \frac{400}{\sqrt{3}} \text{ feet}$$

$$\text{Height of the tower} = AB = AD + DB$$

$$= \frac{400}{\sqrt{3}} + 400 = 400\left(1 + \frac{1}{\sqrt{3}}\right) \text{ feet}$$

## 12.3 EXERCISES

**Real-World Applications**

**1.** A 33-ft ladder leans against a building so that the angle between the ground and the ladder is 60°. How high does the ladder reach up the side of the building?
**2.** A radio tower is located 325 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 60ř, and that the angle of depression to the bottom of the tower is 30ř. How tall is the tower?
**3.** The angle of elevation to the top of a building in New York is found to be 30 degrees from the ground at a distance of 25 meter from the base of the building. Using this information, find the height of the building.
**4.** From the top of a building of height 200 feet, a person determines that the angle of depression of a stone on the ground is 60 degrees. How far is the stone from the base of the building?

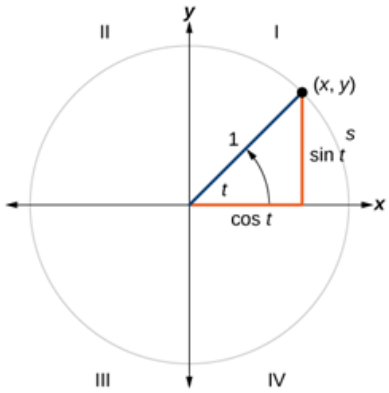
| Angle $\theta$ | | Functions | | |
|---|---|---|---|---|
| Degrees | Radians | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
| 0° | 0 | 0 | 1 | 0 |
| 30° | $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}}$ |
| 45° | $\frac{\pi}{4}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | 1 |
| 60° | $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| 90° | $\frac{\pi}{2}$ | 1 | 0 | $\infty$ |
| 270° | $\frac{3\pi}{2}$ | $-1$ | 0 | $\infty$ |
| 360° | $2\pi$ | 0 | 1 | 0 |

## 12.4 Circular Functions

To define our trigonometric functions, we begin by drawing a **unit circle**, a circle centered at the origin with radius 1. The angle (in radians) that $t$ intercepts forms an arc of length s. Using the formula $s = rt$, and knowing that $r = 1$, we see that for a unit circle, $s = t$

Recall that the $x$ and $y$ axes divide the coordinate plane into four quarters called quadrants. We label these quadrants to mimic the direction a positive angle would sweep. The four quadrants are labeled I, II, III, and IV.

For any angle $t$, we can label the intersection of the terminal side and the unit circle as by its coordinates, $(x, y)$. The coordinates $x$ and $y$ will be the outputs of the trigonometric functions $f(t) = \cos t$ and $f(t) = \sin t$ respectively. This means $x = \cos t$ and $y = \sin t$.




### Unit Circle

A unit circle has a center at (0,0) and radius 1. In a unit circle, the length of the intercepted arc is equal to the radian measure of the central angle t.

Let $(x, y)$ be the endpoint on the unit circle of an arc of arc length s. The $x$ and $y$ coordinates of this point can be described as functions of the angle.

### Sine and Cosine Functions

If $t$ is a real number and a point $(x, y)$ on the unit circle corresponds to an angle of $t$, then

$$\cos t = x$$
$$\sin t = y$$

**Example 1**

If $\sin t = \frac{3}{7}$ and if $t$ is in the second quadrant, find $\cos t$.

**Solution :**

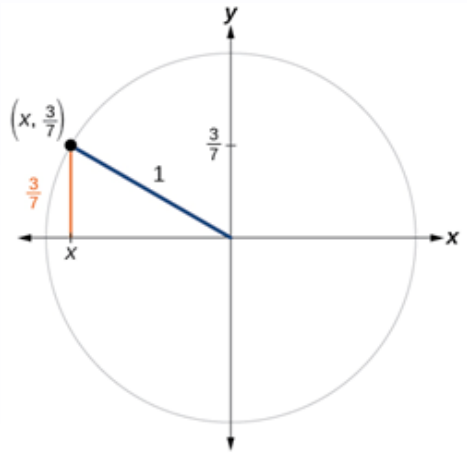
If we drop a vertical line from the point on the unit circle corresponding to t, we create a right triangle, from which we can see that the Pythagorean Identity is simply one case of the Pythagorean Theorem.

Substituting the value of sine into Pythagorean identity,

$$\cos^2 t + \sin^2 t = 1$$
$$\cos^2 t = 1 - \sin^2 t$$
$$= 1 - \frac{9}{49}$$
$$= \frac{40}{49}$$
$$\cos t = \pm\frac{2\sqrt{10}}{7}$$

Since the angle is in the second quadrant, $x$-value is a negative real number and hence cosine is also negative.

$$\therefore \cos t = -\frac{2\sqrt{10}}{7}$$




We have the cosine and sine values for all of the most commonly encountered angles in the first quadrant of the unit circle are summarized as follows.

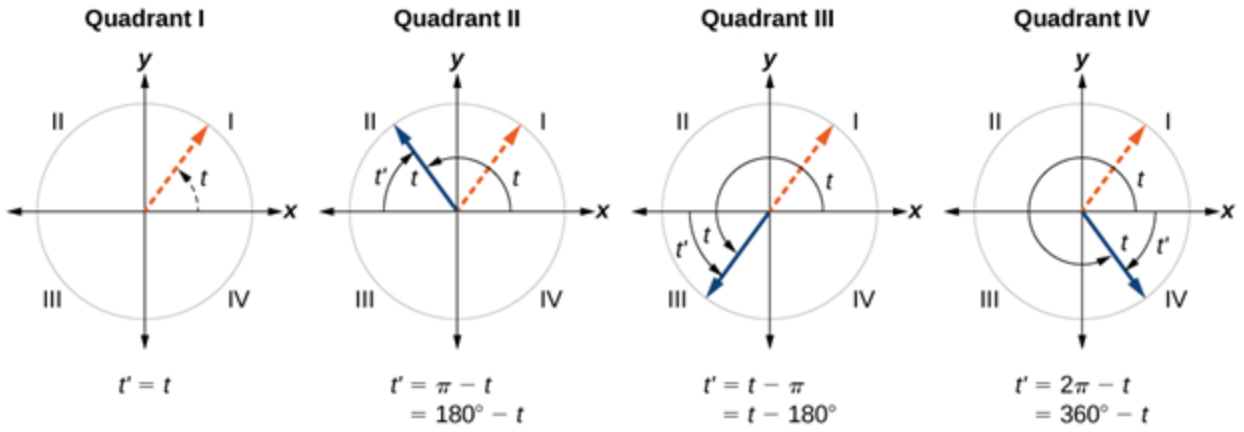
| $\theta$ | $0^0 = 0$ | $30° = \frac{\pi}{6}$ | $45° = \frac{\pi}{4}$ | $60° = \frac{\pi}{3}$ | $90° = \frac{\pi}{2}$ |
|---|---|---|---|---|---|
| $\sin\theta$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| $\cos\theta$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 |

**Reference Angles**

For any given angle in the first quadrant, there is an angle in the second quadrant with the same sine value. Because the sine value is the $y$ -coordinate on the unit circle, the other angle with the same sine will share the same $y$ -value, but have the opposite $x$ -value. Therefore, its cosine value will be the opposite of the first angle's cosine value.

Likewise, there will be an angle in the fourth quadrant with the same cosine as the original angle. The angle with the same cosine will share the same $x$ -value but will have the opposite $y$ value. Therefore, its sine value will be the opposite of the original angle's sine value.

An angle's reference angle is the acute angle; $t$, formed by the terminal side of the angle $t$ and the horizontal axis. A reference angle is always an angle between 0° and 90°, or 0 and $\frac{\pi}{2}$ radians. As we can see from Figure for any angle in quadrants II, III or IV, there is a reference angle in quadrant I.




**Example 2**

Find the coordinates of the point on a unit circle corresponding to an angle of $\frac{7\pi}{6}$.

**Solution :**

$\frac{7\pi}{6}$ is in the $3^{\text{rd}}$ quadrant.

Find the reference angle $\frac{7\pi}{6} - \pi = \frac{\pi}{6}$

Now $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

Since the original angle is in the 3 rd quadrant, both $x$ and $y$ are negative.

$$\therefore \cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2} \text{ and } \sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$$

$\therefore$ The coordinates of the point are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the unit circle.

## 12.4 EXERCISES

Answer the following questions:

(1) If $\cos t = \frac{1}{7}$ and $t$ is in the $4^{\text{th}}$ quadrant, find all the other trigonometric ratios.

(2) If $\cos t = \frac{2}{9}$ and $t$ is in the 1 st quadrant, find all the other trigonometric ratios.

(3) If $\sin t = \frac{3}{8}$ and $t$ is in the $2^{\text{nd}}$ quadrant, find all the other trigonometric ratios.

(4) If $\sin t = -\frac{1}{4}$ and $t$ is in the $3^{\text{rd}}$ quadrant, find all the other trigonometric ratios.

(5) Find the coordinates of the point on a unit circle corresponding to an angle of $220°$.

(6) Find the coordinates of the point on a unit circle corresponding to an angle of $120°$.

(7) Find the coordinates of the point on a unit circle corresponding to an angle of $\frac{7\pi}{4}$.

(8) Find the coordinates of the point on a unit circle corresponding to an angle of $\frac{5\pi}{9}$.

## 12.5 Sum and Difference Identities

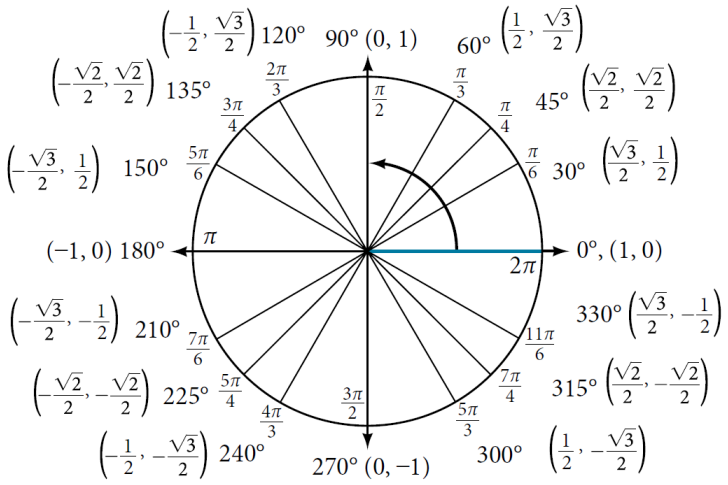
How can the height of a mountain be measured? What about the distance from Earth to the sun? Like many seemingly impossible problems, we rely on mathematical formulas to find the answers. The trigonometric identities, commonly used in mathematical proofs, have had real-world applications for centuries, including their use in calculating long distances.

The trigonometric identities we will examine in this section can be traced to a Persian astronomer who lived around 950 AD, but the ancient Greeks discovered these same formulas much earlier and stated them in terms of chords. These are special equations or postulates, true for all values input to the equations, and with innumerable applications.

In this section, we will learn techniques that will enable us to solve problems such as the ones presented above. The formulas that follow will simplify many trigonometric expressions and equations. Keep in mind that, throughout this section, the term formula is used synonymously with the word identity.

### Using the Sum and Difference Formulas

Finding the exact value of the sine, cosine, or tangent of an angle is often easier if we can rewrite the given angle in terms of two angles that have known trigonometric values. We can use the special angles, which we can review in the unit circle shown




The sum and difference formulas for sine, cosine, and tan so that we can find the sine, cosine, and tan of a given angle respectively if we can break it up into the sum or difference of two of the special angles.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$
$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

**Example 1**

Using the formula for the cosine of the difference of two angles, find the exact value of

$$\cos\left(\frac{5\pi}{4} - \frac{\pi}{6}\right)$$

**Solution :**

Begin by writing the formula for the cosine of the difference of two angles. Then

substitute the given values.

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\cos\left(\frac{5\pi}{4} - \frac{\pi}{6}\right) = \cos\left(\frac{5\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{5\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$$

$$= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$

$$= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

$$= \frac{-\sqrt{6} - \sqrt{2}}{4}$$

## Example 2

Find the exact value of $\cos(75°)$.

## Solution :

As $75° = 45° + 30°$,

we can evaluate $\cos(75°)$ as $\cos(45° + 30°)$.

Let's begin by writing the formula and substitute the given angles.
$\alpha = 45°$ and $\beta = 30°$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(45° + 30°) = \cos(45°)\cos(30°) - \sin(45°)\sin(30°)$$

$$= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

## Example 3

Use the sum and difference identities to evaluate $\sin(15°)$

## Solution :

As $15° = 45° - 30°$,

we can evaluate $\sin(15°)$ as $\sin(45° - 30°)$

Let's begin by writing the formula and substitute the given angles.
$\alpha = 45°$ and $\beta = 30°$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\sin(45° - 30°) = \sin(45°)\cos(30°) - \cos(45°)\sin(30°)$$

Next, we need to find the values of the trigonometric expressions.

$$\sin(45°) = \frac{\sqrt{2}}{2}, \quad \cos(30°) = \frac{\sqrt{3}}{2},$$

$$\cos(45°) = \frac{\sqrt{2}}{2}, \quad \sin(30°) = \frac{1}{2}$$

Now we can substitute these values into the equation and simplify.

$$\begin{aligned}\sin(45° - 30°) &= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

## Example 4

Find the exact value of $\tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$

## Solution :

Lets first write the sum formula for tangent and substitute the given angles into the formula.

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \frac{\tan\left(\frac{\pi}{6}\right) + \tan\left(\frac{\pi}{4}\right)}{1 - \left(\tan\left(\frac{\pi}{6}\right)\tan\left(\frac{\pi}{4}\right)\right)}$$

Next, we determine the individual tangents within the formulas:

$$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}, \quad \tan\left(\frac{\pi}{4}\right) = 1$$

So we have

$$\begin{aligned}\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right) &= \frac{\frac{1}{\sqrt{3}}+1}{1-\left(\frac{1}{\sqrt{3}}\right)(1)} \\ &= \frac{\frac{1+\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}} \\ &= \frac{1+\sqrt{3}}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}-1}\right) \\ &= \frac{\sqrt{3}+1}{\sqrt{3}-1}\end{aligned}$$

**Example 5**

Use the sum and difference identities to evaluate $\tan(15°)$

**Solution :**

As $15° = 60° - 45°$,

we can evaluate $\tan(15°)$ as $\tan(60° - 45°)$

Let's begin by writing the formula and substitute the given angles.

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

$$\tan(60°-45°) = \frac{\tan(60°)-\tan(45°)}{1+\tan(60°)\tan(45°)}$$

Next, we need to find the values of the trigonometric expressions.

$$\tan(60°) = \sqrt{3},$$
$$\tan(45°) = 1$$

Now we can substitute these values into the equation and simplify.

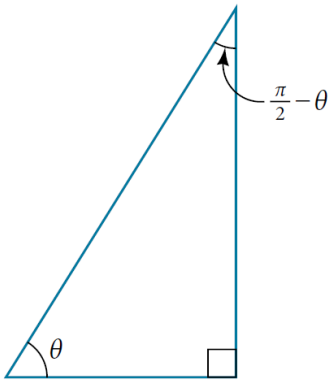
$$\tan(60°-45°) = \frac{\sqrt{3}-1}{1+\sqrt{3}}$$

$$\tan(15°) = \frac{\sqrt{3}-1}{1+\sqrt{3}\times 1}$$

$$\tan(15°) = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

**Sum and Difference Formulas for Cofunctions**

Now that we can find the sine, cosine, and tangent functions for the sums and differences of angles, we can use them to do the same for their cofunctions. You may recall from Right Triangle Trigonometry that, if the sum of two positive angles is $\frac{\pi}{2}$, those two angles are complements, and the sum of the two acute angles in a right triangle is $\frac{\pi}{2}$, so they are also complements. In Figure, notice that if one of the acute angles is labeled as $\theta$, then the other acute angle must be labeled $\left(\frac{\pi}{2}-\theta\right)$.




Notice also that $\sin\theta = \cos\left(\frac{\pi}{2}-\theta\right)$: opposite over hypotenuse. Thus, when two angles are complimentary, we can say that the sine of $\theta$ equals the cofunction of the complement of $\theta$. Similarly, tangent and cotangent are cofunctions, and secant and cosecant are cofunctions.

From these relationships, the cofunction identities are formed. Recall that you first encountered these identities in the Unit Circle: Sine and Cosine Functions.

The cofunction identities are summarized

$$\begin{array}{lll} \sin\theta = \cos\left(\frac{\pi}{2}-\theta\right) & \cos\theta = \sin\left(\frac{\pi}{2}-\theta\right) & \tan\theta = \cot\left(\frac{\pi}{2}-\theta\right) \\ \sec\theta = \csc\left(\frac{\pi}{2}-\theta\right) & \csc\theta = \sec\left(\frac{\pi}{2}-\theta\right) & \cot\theta = \tan\left(\frac{\pi}{2}-\theta\right) \end{array}$$

**Example 6**

Write $\tan\left(\frac{\pi}{9}\right)$ in terms of its cofunction.

**Solution :**

The cofunction of $\tan\theta = \cot\left(\frac{\pi}{2}-\theta\right)$. Thus,

$$\begin{aligned} \tan\left(\frac{\pi}{9}\right) &= \cot\left(\frac{\pi}{2}-\frac{\pi}{9}\right) \\ &= \cot\left(\frac{9\pi}{18}-\frac{2\pi}{18}\right) \\ &= \cot\left(\frac{9\pi-2\pi}{18}\right) \\ \tan\left(\frac{\pi}{9}\right) &= \cot\left(\frac{7\pi}{18}\right) \end{aligned}$$

## 12.5 EXERCISES

Answer the following questions:

I. Simplify the expression

(1) $\cos\left(\frac{\pi}{2} - x\right)$ (2) $\sin\left(\pi - x\right)$ (3) $\sin\left(\frac{\pi}{3} + x\right)$ (4) $\sin\left(\frac{\pi}{4} - x\right)$

(5) $\cos\left(\frac{5\pi}{4} + x\right)$ (6) $\cos\left(\frac{7\pi}{6} + x\right)$ (7) $\tan\left(\frac{\pi}{4} - x\right)$ (8) $\tan\left(\frac{\pi}{4} + x\right)$

II. Find the exact value algebraically, and then confirm the answer with a calculator to the fourth decimal point.

(1) $\sin(75°)$ (2) $\sin(195°)$ (3) $\sin(-15°)$ (4) $\cos(345°)$

(5) $\cos(165°)$ (6) $\cos(15°)$ (7) $\tan(75°)$ (8) $\tan(-15°)$

# Chapter 13

# Trigonometry: Oblique Triangles

## Contents


**13.1 Inverse Trigonometric Functions**

**13.2 Solving Trigonometric Equations**

**13.3 Non-right Triangles: Law of Sines**

13.3.1 Solving Applied Problems using the Law of Sines:

**13.4 Non-right Triangles: Law of Cosines**

13.4.1 Law of Cosines


**Learning outcome covered:**

(w) `Use the laws of sines and cosines to solve a triangle and real life problems.`

### Learning Objectives

**By the end of this chapter, the students will be able to:**

- find the exact value of expressions involving the inverse sine, cosine and tangent functions.
- use a calculator to evaluate inverse trigonometric functions.
- use the law of sines and cosines to solve oblique triangles.
- solve applied problems using the Law of sines and cosines.

# Introduction

For any right triangle, given one other angle and the length of one side, we can figure out what the other angles and sides are. But what if we are given only two sides of a right triangle? We need a procedure that leads us from a ratio of sides to an angle. This is where the notion of an inverse to a trigonometric function comes into play. In this section, we will explore the inverse trigonometric functions. Furthermore, we will learn to solve oblique triangle using laws of sines and cosines.

## 13.1 Inverse Trigonometric Functions

In order to use inverse trigonometric functions, we need to understand that an inverse trigonometric function "undoes" what the original trigonometric function "does", as is the case with any other function and its inverse. In other words, the domain of the inverse function is the range of the original function, and vice versa.

For example, if $f(x) = \sin x$, then we would write $f^{-1}(x) = \sin^{-1} x$.

Be aware that $\sin^{-1} x$ does not mean $\dfrac{1}{\sin x}$.

On the restricted domains, we can define the inverse trigonometric functions.

- The inverse sine function $y = \sin^{-1} x$ means $x = \sin y$. The inverse sine function is sometimes called the arcsine function, and notated $\arcsin x$

$$y = \sin^{-1} x \text{ has domain} [-1, 1] \text{ and range } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

- The inverse cosine function $y = \cos^{-1} x$ means $x = \cos y$. The inverse cosine function is sometimes called the arccosine function, and notated $\arccos x$.

$$y = \cos^{-1} x \text{ has domain } [-1, 1] \text{ and range } [0, \pi]$$

- The inverse tangent function $y = \tan^{-1} x$ means $x = \tan y$. The inverse tangent function is sometimes called the arctangent function, and notated $\arctan x$.

$$y = \tan^{-1} x \,\text{has domain} (-\infty, \infty) \text{ and range } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

**Relations for inverse sine, cosine and tangent functions**

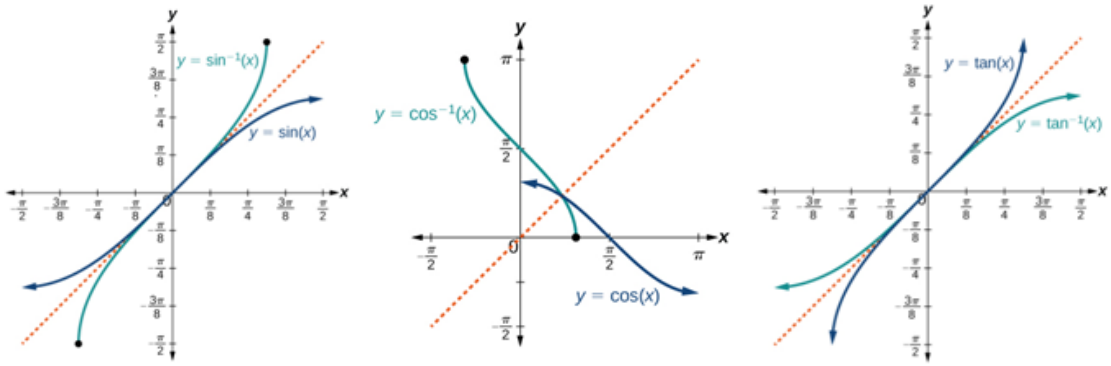
For angles in the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, if $\sin y = x$, then $\sin^{-1} x = y$.

For angles in the interval $[0, \pi]$, if $\cos y = x$, then $\cos^{-1} x = y$.

For angles in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if $\tan y = x$, then $\tan^{-1} x = y$.

The graphs of the inverse functions are shown in figure given below. Notice that the output of each of these inverse functions is a number, an angle in radian measure.

To find the domain and range of inverse trigonometric functions, switch the domain and range of the original functions. Each graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.




**Finding the Exact Value of Expressions Involving the Inverse Sine, Cosine, and Tangent Functions**

Now that we can identify inverse functions, we will learn to evaluate them. For most values in their domains, we must evaluate the inverse trigonometric functions by using a calculator, interpolating from a table, or using some other numerical technique. Just as we did with the original trigonometric functions, we can give exact values for the inverse functions when we are using the special angles, specifically $\frac{\pi}{6}\,(30°)$, $\frac{\pi}{4}\,(45°)$, and $\frac{\pi}{3}\,(60°)$, and their reflections into other quadrants.

Given a “special" input value, evaluate an inverse trigonometric function.

1. Find angle $x$ for which the original trigonometric function has an output equal to the given input for the inverse trigonometric function.
2. If $x$ is not in the defined range of the inverse, find another angle $y$ that is in the defined range and has the same sine, cosine, or tangent as $x$ depending on which corresponds to the given inverse function.

## Evaluating Inverse Trigonometric Functions for Special Input Values

**Example 1**

Evaluate each of the following.

(a) $\sin^{-1}\left(\frac{1}{2}\right)$ (b) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ (c) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ (d) $\tan^{-1}(1)$

**Solution :**

(a) Evaluating $\sin^{-1}\left(\frac{1}{2}\right)$ is the same as determining the angle that would have a sine value of $\frac{1}{2}$ In other words, what angle $x$ would satisfy $\sin x = \left(\frac{1}{2}\right)$? There are multiple values that would satisfy this relationship, such as $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ but we know we need the angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so the answer will be $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$. Remember that the inverse is a function, so for each input, we will get exactly one output.

(b) To evaluate $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, we know that $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ both have a sine value of $-\frac{\sqrt{2}}{2}$, but neither is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. For that, we need the negative angle coterminal with $\frac{7\pi}{4}$ : $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$.

(c) To evaluate $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$,we are looking for an angle in the interval $[0, \pi]$ with a cosine value of $-\frac{\sqrt{3}}{2}$. The angle that satisfies this is $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$.

(d) Evaluating $\tan^{-1}(1)$ we are looking for an angle in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with a tangent value of 1. The correct angle is $\tan^{-1}(1) = \frac{\pi}{4}$

### Using a Calculator to Evaluate Inverse Trigonometric Functions

To evaluate inverse trigonometric functions that do not involve the special angles discussed previously, we will need to use a calculator or other type of technology. Most scientific calculators and calculator-emulating applications have specific keys or buttons for the inverse sine, cosine, and tangent functions.

## 13.1 EXERCISES

I. Evaluate exact values for each of the following.

(1) $\tan^{-1}(-1)$ (2) $\sin^{-1}(-1)$ (3) $\cos^{-1}(-1)$ (4) $\cos^{-1}\left(\frac{1}{2}\right)$

II. Evaluate exact values for each of the following.

(1) $\sin^{-1}(0.97)$ (2) $\cos^{-1}(-0.4)$ (3) $\tan^{-1}(6)$ (4) $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$

(5) $\cos\left(\tan^{-1}\left(\frac{5}{12}\right)\right)$

## 13.2 Solving Trigonometric Equations

Trigonometric equations are, as the name implies, equations that involve trigonometric functions. Similar in many ways to solving polynomial equations or rational equations, only specific values of the variable will be solutions, if there are solutions at all. Often we will solve a trigonometric equation over a specified interval. However, just as often, we will be asked to find all possible solutions, and as trigonometric functions are periodic, solutions are repeated within each period. In other words, trigonometric equations may have an infinite number of solutions. Additionally, like rational equations, the domain of the function must be considered before we assume that any solution is valid. The period of both the sine function and the cosine function is $2\pi$. In other words, every $2\pi$ units, the $y-$values repeat. If we need to find all possible solutions, then we must add $2\pi k$, where $k$ is an integer, to the initial solution. Recall the rule that gives the format for stating all possible solutions for a function where the period is $2\pi$:

$$\sin\ = \sin(\ \pm 2k\pi)$$

There are similar rules for indicating all possible solutions for the other trigonometric functions. Solving trigonometric equations requires the same techniques as solving algebraic equations. We read the equation from left to right, horizontally, like a sentence. We look for known patterns, factor, find common denominators, and substitute certain expressions with a variable to make solving a more straightforward process. However, with trigonometric equations, we also have the advantage of using the identities we developed in the previous sections.

**Example 1**

Find all possible exact solutions for the equation $\cos\theta = \frac{1}{2}$.

**Solution :**

From the unit circle, we know that

$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

These are the solutions in the interval $[0, 2\pi]$.

All possible solutions are given by

$$\theta = \frac{\pi}{3} \pm 2k\pi \text{ and } \theta = \frac{5\pi}{3} \pm 2k\pi$$

where $k$ is an integer.

**Example 2**

Find all possible exact solutions for the equation $\sin t = \frac{1}{2}$.

**Solution :**

Solving for all possible values of $t$ means that solutions include angles beyond the period of $2\pi$.

The solutions are $t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$.

But the problem is asking for all possible values that solve the equation.

Therefore, the answer is

$$t = \frac{\pi}{6} \pm 2\pi k \quad \text{and} \quad t = \frac{5\pi}{6} \pm 2\pi k$$

where $k$ is an integer.

**Example 3**

Solve the equation exactly: $2\cos\theta - 3 = -5, \quad 0 \leq \theta < 2\pi$.

**Solution :**

Use algebraic techniques to solve the equation.

$$2\cos\theta - 3 = -5$$
$$2\cos\theta = -2$$
$$\cos\theta = -1$$
$$\theta = \pi$$

**Example 4**

Solve the problem exactly: $2\sin^2\theta - 1 = 0, 0 \le \theta < 2\pi$.

**Solution :**

As this problem is not easily factored, we will solve using the square root property. First, we use algebra to isolate $\sin\theta$. Then we will find the angles.

$$\begin{aligned} 2\sin^2\theta - 1 &= 0 \\ 2\sin^2\theta &= 1 \\ \sin^2\theta &= \frac{1}{2} \\ \sqrt{\sin^2\theta} &= \pm\sqrt{\frac{1}{2}} \\ \sin\theta &= \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2} \\ \theta &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \end{aligned}$$

**Example 5**

Solve the following equation exactly: $\csc\theta = -2, 0 \le \theta < 4\pi$.

**Solution :**

We want all values of $\theta$ for which $\csc\theta = -2$ over the interval $0 \le \theta < 4\pi$.

$$\begin{aligned} \csc\theta &= -2 \\ \frac{1}{\sin\theta} &= -2 \\ \sin\theta &= -\frac{1}{2} \\ \theta &= \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6} \end{aligned}$$

**Example 6**

Solve the equation quadratic in form exactly: $2\sin^2\theta - 3\sin\theta + 1 = 0, 0 \le \theta < 2\pi$.

**Solution :**

We can factor using grouping. Solution values of $\theta$ can be found on the unit circle:

$$\begin{aligned}(2\sin\theta - 1)(\sin\theta - 1) &= 0\\ 2\sin\theta - 1 &= 0\\ \sin\theta &= \frac{1}{2}\\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6}\\ \sin\theta &= 1\\ \theta &= \frac{\pi}{2}\end{aligned}$$

**Example 7**

Solve the equation exactly: $\cos^2\theta + 3\cos\theta - 1 = 0, \quad 0 \le \theta < 2\pi.$

**Solution :**

We begin by using substitution and replacing $\cos\theta$ with $x$. It is not necessary to use substitution, but it may make the problem easier to solve visually. Let $\cos\theta = x$. We have

$$x^2 + 3x - 1 = 0$$

The equation cannot be factored, so we will use the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Replace $x$ with $\cos\theta$, and solve. Thus,

$$\begin{aligned}x &= \frac{-3 \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2}\\ &= \frac{-3 \pm \sqrt{13}}{2}\\ \cos\theta &= \frac{-3 \pm \sqrt{13}}{2}\\ \theta &= \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right)\end{aligned}$$

Note that only the $+$ sign is used. This is because we get an error when we solve $\theta = \cos^{-1}\left(\dfrac{-3 - \sqrt{13}}{2}\right)$ on calculator, since the domain of the inverse cosine function is $[-1, 1]$.

However, there is a second solution:

$$\theta = \cos^{-1}\left(\frac{-3+\sqrt{13}}{2}\right)$$

$$\approx 1.26$$

This terminal side of the angle lies in quadrant I. Since cosine is also positive in quadrant IV, the second solution is

$$\theta = 2\pi - \cos^{-1}\left(\frac{-3+\sqrt{13}}{2}\right)$$

$$\approx 5.02$$

**Example 8**

Solve the equation exactly: $2\sin^2\theta - 5\sin\theta + 3 = 0, \quad 0 \le \theta \le 2\pi.$

**Solution :**

Using grouping, this quadratic can be factored. Either make the real substitution, $\sin\theta = u$, or imagine it, as we factor:

$$2\sin^2\theta - 5\sin\theta + 3 = 0$$
$$(2\sin\theta - 3)(\sin\theta - 1) = 0$$

Now set each factor equal to zero.

$$2\sin\theta - 3 = 0$$
$$2\sin\theta = 3$$
$$\sin\theta = \frac{3}{2}$$
$$\sin\theta - 1 = 0$$
$$\sin\theta = 1$$

Next solve for $\theta$ : $\sin\theta \neq \frac{3}{2}$, as the range of the sine function is $[-1, 1]$. However, $\sin\theta = 1$, giving the solution $\theta = \frac{\pi}{2}$.

## 13.2 EXERCISES

Answer the following questions:

I. Find all solutions exactly on the interval $0 \leq \theta < 2\pi$.

(1) $2\sin\theta = -\sqrt{2}$ (2) $2\sin\theta = \sqrt{3}$ (3) $2\cos\theta = 1$

(4) $2\cos\theta = -\sqrt{2}$ (5) $\tan\theta = -1$ (6) $\tan x = 1$

(7) $\cot x + 1 = 0$ (8) $4\sin^2 x - 2 = 0$ (9) $\csc^2 x - 4 = 0$

II. Solve exactly on the interval $[0, 2\pi)$. Use the quadratic formula if the equations do not factor.

(1) $\tan^2 x - \sqrt{3}\tan x = 0$ (2) $\sin^2 x + \sin x - 2 = 0$ (3) $\sin^2 x - 2\sin x - 4 = 0$

(4) $5\cos^2 x + 3\cos x - 1 = 0$ (5) $3\cos^2 x - 2\cos x - 2 = 0$ (6) $5\sin^2 x + 2\sin x - 1 = 0$

(7) $\tan^2 x + 5\tan x - 1 = 0$ (8) $-\tan^2 x - \tan x - 2 = 0$

## 13.3 Non-right Triangles: Law of Sines

Any triangle that is not a right triangle is an oblique triangle. Solving an oblique triangle means finding the measurements of all three angles and all three sides. To do so, we need to start with at least three of these values, including at least one of the sides. We will investigate three possible oblique triangle problem situations:

1. **ASA (angle-side-angle)**

We know the measurements of two angles and the included side.

2. **AAS (angle-angle-side)**

We know the measurements of two angles and a side that is not between the known angles.

3. **SSA (side-side-angle)**

We know the measurements of two sides and an angle that is not between the known sides.

Here we get three different possibilities

- No solution, if $|\sin x| > 1$ : no such triangle exists.
- Two solutions, if $|\sin x| < 1$ and obtuse angle exits.
  (i.e. $\sin x$ and $\sin(180^\circ - x)$ exits ) : Two different triangles exist.
- One solution, if $|\sin x| < 1$ and obtuse angle does not exit: Exactly one triangle exists.

Knowing how to approach each of these situations enables us to solve oblique triangles without having to drop a perpendicular to form two right triangles. Instead, we can use the fact that the ratio of the measurement of one of the angles to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. Let's see how this statement is derived by considering the triangle shown in figure below.

Using the right triangle relationships, we know that

$$\sin\alpha = \frac{h}{b} \text{ and } \sin\beta = \frac{h}{a}$$

Solving both equations gives two different expressions for $h$.

$$h = b\sin\alpha \text{ and } h = a\sin\beta$$

We then set the expressions equal to each other.

$$b\sin\alpha = a\sin\beta$$

$$\left(\frac{1}{ab}\right)(b\sin\alpha) = (a\sin\beta)\left(\frac{1}{ab}\right)$$

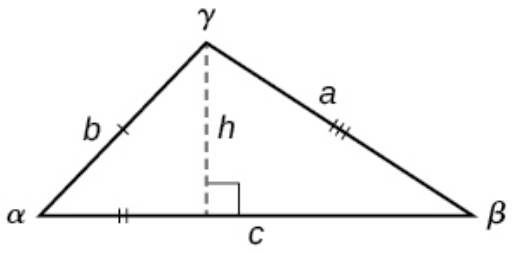
Multiply both sides by $\frac{1}{ab}$,

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$$

Similarly, we can compare the other ratios.

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \text{ and } \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Collectively, these relationships are called the Law of sines.




$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

**Law of Sines**

It states that the ratio of the measurement of one angle of a triangle to the length of its opposite side is equal to the remaining two ratios of angle measure to opposite side; any pair of proportions may be used to solve for a missing angle or side.

The other forms of the law of sines are

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

and

$$\frac{a}{b} = \frac{\sin\alpha}{\sin\beta}, \qquad \frac{a}{c} = \frac{\sin\alpha}{\sin\gamma}, \qquad \frac{b}{c} = \frac{\sin\beta}{\sin\gamma}$$

Note the standard way of labeling triangles: angle $\alpha$ (alpha) is opposite side $a$; angle $\beta$ (beta) is opposite side $b$; and angle $\gamma$ (gamma) is opposite side $c$ See figure above.

While calculating angles and sides, be sure to carry the exact values through to the final answer. Generally, final answers are rounded to the nearest tenth, unless otherwise specified.

**Points to remember:**

- sum of the angles in a triangle is $180°$
- range of sine is $[-1, 1]$
- $\sin x = \sin(180° - x)$

## Example 1

**ASA (angle-side-angle)** case:

Given $\angle A = 75°$ and $\angle C = 42°, b = 22$ feet. Solve the triangle.

## Solution :

Sum of the interior angles,

$$\begin{aligned} \angle A + \angle B + \angle C &= 180° \\ \angle B &= 180° - \angle A - \angle C \\ &= 180° - 75° - 42° \\ &= 63° \end{aligned}$$

We know

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin 75°}{a} = \frac{\sin 63°}{22} = \frac{\sin 42°}{c}$$

$$a = \frac{22 \sin 75°}{\sin 63°} \approx 23.85 \quad \& \quad c = \frac{22 \sin 42°}{\sin 63°} \approx 16.52$$

$$\therefore \angle B = 63°, \quad a \approx 23.85 \text{ feet}, c \approx 16.52 \text{ feet}$$

## Example 2

**AAS (angle-angle-side)** case:

Given $\angle B = 45°$ and $\angle C = 30°, c = 2$. Solve the triangle.

## Solution :

Sum of the interior angles,

$$\begin{aligned} \angle A + \angle B + \angle C &= 180° \\ \angle B &= 180° - \angle A - \angle C \\ &= 180° - 30° - 45° \\ &= 105° \end{aligned}$$

We know

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$
$$\frac{\sin 105}{a} = \frac{\sin 45^\circ}{b} = \frac{\sin 30^\circ}{2}$$
$$a = \frac{2\sin 105^\circ}{\sin 30^\circ} \approx 3.86 \quad \& \quad b = \frac{2\sin 45^\circ}{\sin 30^\circ} \approx 2.82$$
$$\therefore \angle A = 105^\circ, \quad a \approx 3.86, \quad b \approx 2.82$$

## Example 3

**SSA (side-side-angle)**: Ambiguous Case

Solve the triangle, given $c = 5, a = 10$ and $\angle C = 45^\circ$

## Solution :

We know

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$
$$\frac{\sin A}{10} = \frac{\sin 45}{5}$$
$$\sin A = \frac{10\sin 45}{5} = \sqrt{2}$$

Which is impossible (i.e. for any angle $A$, $|\sin A| \leq 1$ ). No such triangle exits. Thus, there is no solution.

## Example 4

**Example on one solution:**

Solve the triangle, given $\angle A = 83^\circ, a = 7.8$, and $b = 7$

## Solution :

we know

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$
$$\frac{\sin B}{7} = \frac{\sin 83^\circ}{7.8}$$
$$\sin B = \frac{7\sin 83^\circ}{7.8}$$
$$= 0.89$$
$$\angle B = \sin^{-1} 0.89 = 62.87^\circ$$

Another possible solution for $\angle B$ is $180° - 62.87° = 117.13°$ (which is impossible as rule for the sum of the interior angle fails)
If $\angle B = 62.87°$ then

$$\angle C = 180° - \angle A - \angle B$$
$$\angle C = 180° - 83° - 62.87° = 34.13°$$
$$\frac{\sin 34.13}{c} = \frac{\sin 83}{7.8}$$
$$c = \frac{7.8 \sin 34.13°}{\sin 83°}$$
$$c = 4.4$$

## Example 5

**Example on two solutions:**

Solve the triangle, given $\angle A = 35°, a = 6$, and $b = 8$

## Solution :

we know

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$
$$\frac{\sin B}{8} = \frac{\sin 35°}{6}$$
$$\sin B = \frac{8 \sin 35°}{6}$$
$$= 0.7648$$
$$\angle B = \sin^{-1} 0.7648$$
$$= 49.9°$$

Another possible solution for $\angle B$ is $180° - 49.9° = 130.1°$

**Case 1:** If $\angle B = 49.9°$ then

$$\angle C = 180° - \angle A - \angle B$$
$$\angle C = 180° - 35° - 49.9° = 95.1°$$
$$\frac{\sin 95.1°}{c} = \frac{\sin 35°}{6}$$
$$c = \frac{6 \sin 95.1°}{\sin 35°}$$
$$c = 10.4$$

**Case 2:** If $\angle B = 130.1°$ then

$$\angle C = 180° - \angle A - \angle B$$
$$\angle C = 180° - 35° - 130.1° = 14.9°$$
$$\frac{\sin 14.9°}{c} = \frac{\sin 35°}{6}$$
$$c = \frac{6 \sin 14.9°}{\sin 35°}$$
$$c = 2.69$$

These are the two possible solutions. Thus, two different triangles exist.

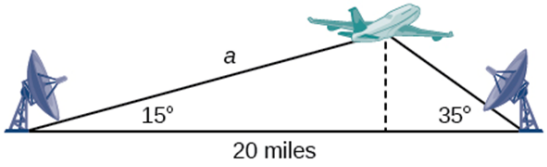
### 13.3.1 Solving Applied Problems using the Law of Sines:

The more we study trigonometric applications, the more we discover that the applications are countless. Some are flat, diagram-type situations, but many applications in calculus, engineering, and physics involve three dimensions and motion.

**Finding the altitude:**

**Example 6**

Find the altitude of the aircraft in the problem introduced at the beginning of this section, shown in figure. Round the altitude to the nearest tenth of a mile.




**Solution :**

To find the elevation of the aircraft, we first find the distance from one station to the aircraft, such as the side $a$, and then use right triangle relationships to find the height of the aircraft, $h$.

Because the angles in the triangle add up to 180 degrees, the unknown angle must be $180° - 15° - 35° = 130°$.

This angle is opposite the side of length 20 miles, allowing us to set up a Law of sines

relationship.

$$\frac{\sin 130^\circ}{20} = \frac{\sin 35^\circ}{a}$$
$$a = \frac{20 \sin 35^\circ}{\sin 135^\circ}$$
$$= 16.22 \text{ miles}$$

The distance from one station to the aircraft is about 16.22 miles.

Now that we know $a$, we can use right triangle relation to solve for $h$.

$$\sin 15^\circ = \frac{opp}{hyp}$$
$$= \frac{h}{a}$$
$$= \frac{h}{16.22}$$
$$h = 16.22 \times \sin 15^\circ$$
$$h = 4.19$$

$\therefore$ The aircraft is at an altitude of approximately 4.2 miles.

**Note:**

If given three angles i.e. AAA is known, it is impossible to find the sides of triangle.

## 13.3 EXERCISES

I. Answer the following questions:

(1) Given $\angle A = 56°, \angle C = 20°$ and $c = 1.7$ m. Find $a$.

(2) Given $\angle A = 98°, \angle C = 43°$ and $b = 22$cm. Find $a$ and $c$

(3) Solve the triangle. Given $\angle A = 80°, a = 120$cm and $b = 121$cm. If there is more than one possible solution, show both.

(4) Solve the triangle if $\angle C = 85°, c = 12$m and $b = 9$m

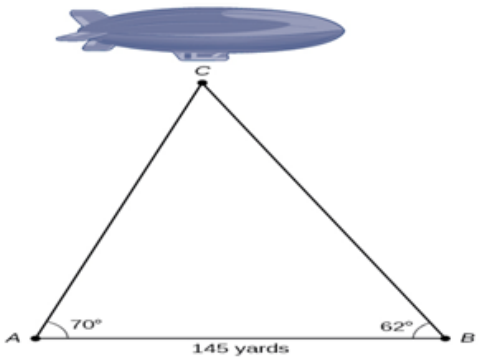
(5) Determine the number of triangle possible. Given $\angle B = 48°, a = 31$ feet and $b = 26$ feet

(6) Find all possible triangles if $\angle A = 50°$, $a = 4$ cm and $b = 10$ cm.

(7) Given $\angle A = 80°, a = 100$miles and $b = 10$miles, find the missing side and angles. If there is more than one possible solution, show both. Round your answer to the nearest tenth.

(8) Solve the triangle, given $\angle A = 63°, c = 4.7$miles and $a = 5.5$miles.

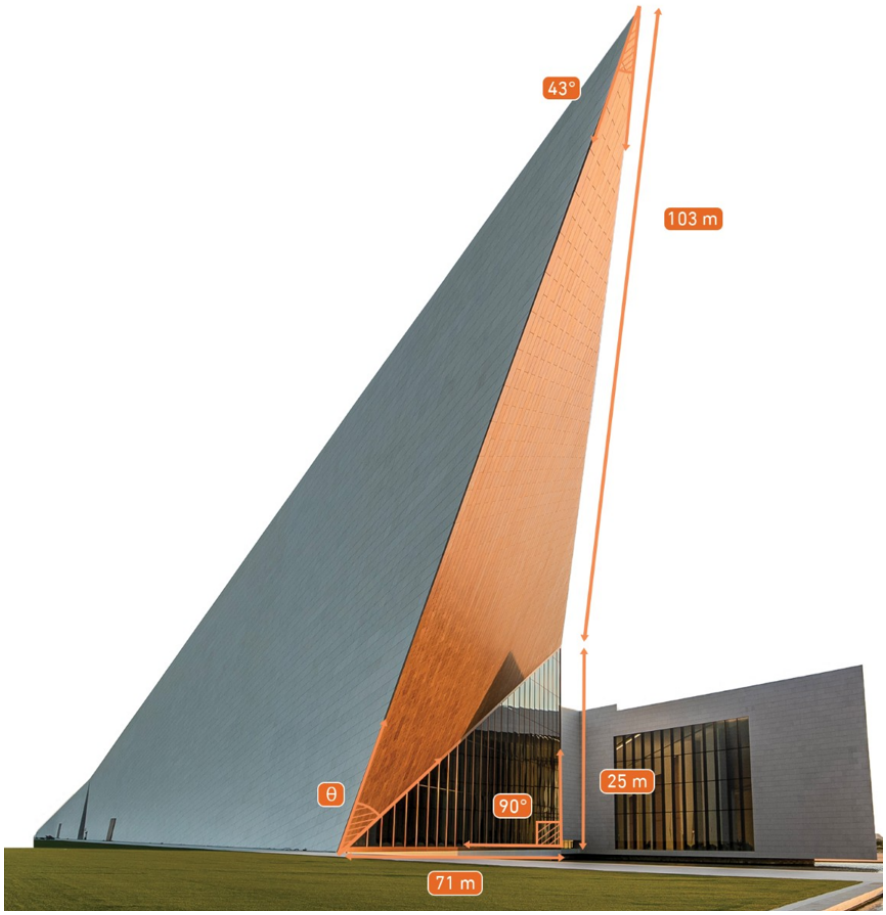
(9) The following diagram represents the height of a blimp flying over a football stadium. Find the height of the blimp if the angle of elevation at the southern end zone, point $A$, is 70°, the angle of elevation from the northern end zone, point B, is 62°, and the distance between the viewing points of the two end zones is 145 yards.



(10) **Oman Across Ages Museum**

Opened on 13 March 2023 in Manah, the Oman Across Ages Museum is an iconic structure that imitates nature; its architectural design was inspired by the surrounding Hajar mountain range.

**Problem.** Refer to the annotated side view in following image. In the right-angled triangle in orange, the horizontal base is 71 m, the vertical side is 25 m, and the slanted side is 103 m.



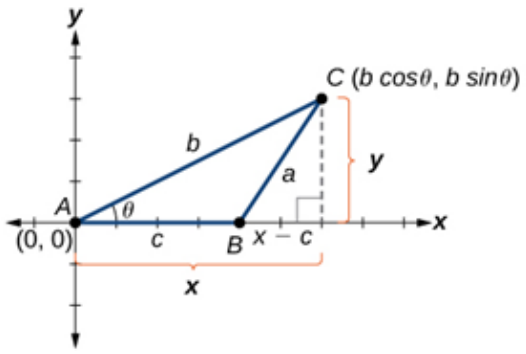

Find the angle $\theta$ (to the nearest degree).

*Note.* Sometimes an image does not accurately reflect the actual dimensions and angles.

## 13.4 Non-right Triangles: Law of Cosines

Unfortunately, while the Law of sines enables us to address many non-right triangle cases, it does not help us with triangles where the known angle is between two known sides, a SAS (side-angle-side) triangle, or when all three sides are known, but no angles are known, a SSS (side-side-side) triangle. In this section, we will investigate another tool, Law of cosines for solving oblique triangles described by these last two cases.

Understanding how the Law of cosines is derived will be helpful in using the formulas. The derivation begins with the Generalized Pythagorean Theorem, which is an extension of the Pythagorean Theorem to non-right triangles. Here is how it works: An arbitrary non-right triangle $ABC$ is placed in the coordinate plane with vertex $A$ at the origin, sidec drawn along the $x$ -axis, and vertex $C$ located at some point $(x, y)$ in the plane, as illustrated in figure. Generally, triangles exist anywhere in the plane, but for this explanation we will place the triangle as noted.




We can drop a perpendicular from $C$ to the $x$ -axis (this is the altitude or height). Recalling the basic trigonometric identities, we know that

$$\cos\theta = \frac{x\,(\text{ adjacent }\,)}{b\,(\text{ hypotenuse }\,)} \text{ and } \sin\theta = \frac{y\,(\text{ opposite }\,)}{b\,(\text{ hypotenuse }\,)}$$

In terms of $\theta, x = b\cos\theta$ and $y = b\sin\theta$, the point $(x, y)$ located at $C$ has coordinates $(b\cos\theta, b\sin\theta)$.

Using the side $(x - c)$ as one leg of a right triangle and $y$ as the second leg, we can find the length of hypotenuse $a$ using the Pythagorean Theorem.
Thus,

$$\begin{aligned} a^2 &= (x-c)^2 + y^2 \\ &= (b\cos\theta - c)^2 + (b\sin\theta)^2 \quad (\text{substitute } x = b\cos\theta, \quad y = b\sin\theta) \\ &= \left(b^2\cos^2\theta - 2bc\cos\theta + c^2\right) + b^2\sin^2\theta \\ &= b^2\left(\cos^2\theta + \sin^2\theta\right) + c^2 - 2bc\cos\theta \\ &= b^2 + c^2 - 2bc\cos\theta \end{aligned}$$

The formula derived is one of the three equations of the Law of cosines. The other equations are found in a similar fashion.

Keep in mind that it is always helpful to sketch the triangle when solving for angles or

sides. In a real-world scenario, try to draw a diagram of the situation. As more information emerges, the diagram may have to be altered. Make those alterations to the diagram and, in the end, the problem will be easier to solve.

### 13.4.1 Law of Cosines

The Law of cosines states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle. For triangles with angles $A, B$ and $C$ and opposite corresponding sides $a, b$ and $c$ respectively, the Law of cosines is given as three equations.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B \quad \text{and}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

To solve for a missing side measurement, the corresponding opposite angle measure is needed.

When solving for an angle, the corresponding opposite side measure is needed. We can use another version of the Law of cosines to solve for an angle.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \quad \text{and}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

We have to use Laws of Cosines to investigate the following two possible oblique triangle problem situations:

**SAS (side-angle-side)**

Given two sides and the angle between them (SAS), find the measures of the remaining side and angles of a triangle.

**Procedure 1:**
Step 1: Using Law of cosines, find unknown side.
Step 2: Using Law of sines, find the angle opposite to the smaller side.
Step 3: Find the unknown angle, using the rule sum of the interior angles in a triangle is 180°

**Procedure 2:**

Step 1: Find any angle, using Law of Cosines.
Step 2: Calculate another angle using again Law of Cosines.
Step 3: Find the unknown angle, using the rule sum of the interior angles in a triangle is.

**Note:**

- Since the inverse cosine can return any angle between 0 and 180 degrees, there will not occur any ambiguous cases while using Law of Cosines.
- It is impossible to estimate the sides of a triangle, if AAA is known, i.e. three angles are known.

**Example 1**

**Example on SAS (side-angle-side)**:

Find the unknown side and angles of the triangle from the known side, angle and side: $\angle B = 30°$, $a = 10$, $c = 12$.

**Solution :**

First find the unknown side $b$ using

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos B \\ b^2 &= 10^2 + 12^2 - 2(10)(12)\cos 30° \\ &= 100 + 144 - 240\left(\frac{\sqrt{3}}{2}\right) = 36.153 \\ b &= 6.013 \end{aligned}$$

Using Law of sines, find the angle opposite to the smaller side i.e. $\angle A$.

$$\begin{aligned} \frac{\sin A}{a} &= \frac{\sin B}{b} \\ \frac{\sin A}{10} &= \frac{\sin 30°}{6.013} \\ \sin A &= \frac{10\sin 30°}{6.013} \\ A &= \sin^{-1}\left(\frac{10\sin 30°}{6.013}\right) \\ &= 56.3 \end{aligned}$$

Now find the unknown angle, using the rule sum of the interior angles in a triangle is

180°.

$$\begin{aligned}\angle A + \angle B + \angle C &= 180^\circ \\ \angle C &= 180^\circ - \angle A - \angle B \\ &= 180^\circ - 56.3^\circ - 30^\circ \\ &= 93.7^\circ\end{aligned}$$

## Example 2

**Example on SSS (side-side-side):**

Solve the triangle, given $a = 20$, $b = 25$ and $c = 18$

## Solution :

Finding angle $A$, using Law of Cosines,

$$\begin{aligned}a^2 &= b^2 + c^2 - 2bc\cos A \\ 20^2 &= 25^2 + 18^2 - 2(25)(18)\cos A \\ 400 &= 949 - 900\cos A \\ \cos A &= \frac{549}{900} \\ \angle A &= \cos^{-1}(0.61) = 52.4^\circ\end{aligned}$$

Using Law of Sines, find the angle opposite to the smaller side, i.e. $\angle C$

$$\begin{aligned}\frac{\sin A}{a} &= \frac{\sin C}{c} \\ \frac{\sin 52.4^\circ}{20} &= \frac{\sin C}{18} \\ \sin C &= \frac{18 \times \sin 52.4^\circ}{20} \\ \angle C &= \sin^{-1}\left(\frac{18 \times \sin 52.4^\circ}{20}\right) \\ \angle C &= 45.48^\circ\end{aligned}$$

Now find the unknown angle, using the rule sum of the interior angles in a triangle is $180^0$.

$$\begin{aligned}\angle A + \angle B + \angle C &= 180^\circ \\ \angle B &= 180^\circ - \angle A - \angle C \\ &= 180^\circ - 52.4^\circ - 45.48^\circ \\ &= 82.12^\circ\end{aligned}$$

### Applied Problems Using the Law of Cosines

Just as the Law of Sines provided the appropriate equations to solve a number of applications, the Law of Cosines is applicable to situations in which the given data fits the cosine models. We may see these in the fields of navigation, surveying, astronomy, and geometry, just to name a few.

### Using Law of Cosines to Solve a Communication Problem

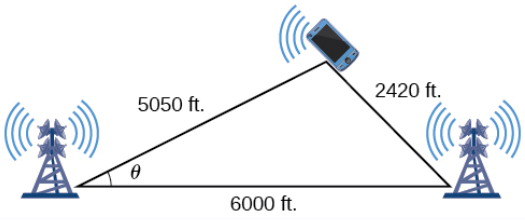
On many cell phones with GPS, an approximate location can be given before the GPS signal is received. This is accomplished through a process called triangulation, which works by using the distances from two known points.

**Example 3**

Suppose there are two cell phone towers within range of a cell phone. The two towers are located 6000 feet apart along a straight highway, running east to west, and the cell phone is north of the highway. Based on the signal delay, it can be determined that the signal is 5050 feet from the first tower and 2420 feet from the second tower. Determine the position of the cell phone north and east of the first tower, and determine how far it is from the highway.

**Solution :**

For simplicity, we start by drawing a diagram and labeling our given information.




Using the Law of Cosines, we can solve for the angle $\theta$. Remember that the Law of cosines uses the square of one side to find the cosine of the opposite angle.

For this example, let $a = 2420\,ft$, $b = 5050\,ft$, and $c = 6000\,ft$.

Thus, $\theta$ corresponds to the opposite side $a = 2420$

$$a^2 = b^2 + c^2 - 2bc\cos\theta$$

$$\cos\theta = \frac{(2420)^2 - (5050)^2 - (6000)^2}{-2(5050)(6000)}$$

$$\cos\theta = 0.9183$$

$$\theta = \cos^{-1}(0.9183)$$

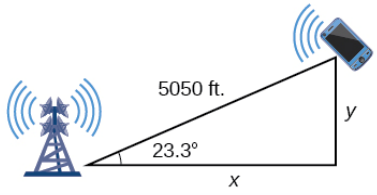
$$\approx 23.3^\circ$$

To answer the questions about the phone's position north and east of the tower, and the distance to the highway, drop a perpendicular from the position of the cell phone, as in figure.

This forms two right triangles, although we only need the right triangle that includes the first tower for this problem.

Using the angle $\theta = 23.3°$ and the basic trigonometric identities, we can find the solutions.
Thus,

$$\cos(23.3°) = \frac{x}{5050}$$
$$x = 5050 \times \cos(23.3°)$$
$$= 4638.15 \quad \text{feet}$$




$$\text{Similarly } \sin(23.3°) = \frac{y}{5050}$$
$$y = 5050 \times \sin(23.3°)$$
$$= 1997.5 \quad \text{feet}$$

Therefore, the cell phone is approximately 4638 feet east and 1998 feet north of the first tower, and 1998 feet from the highway.

## 13.4 EXERCISES

I. Answer the following questions:

(1) Solve the triangle, given $\alpha = 30°$, $b = 12\,\text{cm}$ and $c = 24\,\text{cm}$

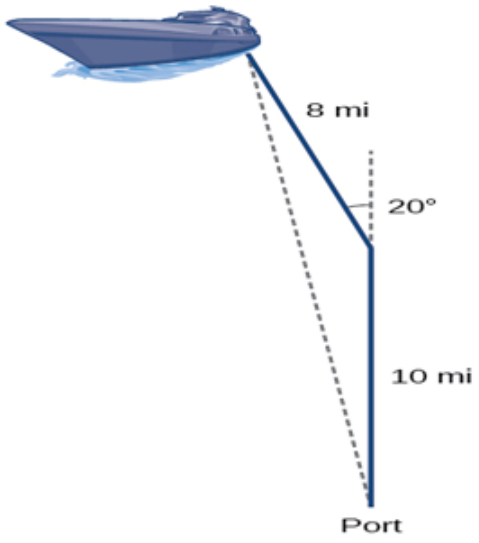
(2) Given $a = 5$ miles, $b = 7$ miles and $c = 10$ miles, find the missing angles.

(3) Solve the triangle: Given $a = 34.6$ feet, $b = 22$ feet and $c = 23.8$ feet

(4) Solve the triangle: Given $a = 4\,km, b = 2\,km$ and $c = 3\,km$

(5) Solve the triangle: Given $a = 6.2\,\text{m}, c = 7.8\,\text{m}$ and $\angle B = 62°$

(6) Suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles. How far from port is the boat? The diagram is here.



# Chapter 14

# Statistics

## Contents


**14.1 Measures of Central Tendency**

**14.2 Summarizing Data into Tables and Graphs**

14.2.1 Representing Data Using Graphs and Charts


**Learning outcome covered:**

(y) Understand basic concepts of descriptive statistics, mean, median, mode and summarize data into tables and simple graphs (bar charts, histogram, and pie chart).

**Learning Objectives**

**By the end of this chapter, the students will be able to:**

- understand the basic concepts of descriptive statistics.
- compute the basic measures of central tendancy.
- summarize a given data in to tables and graphs.

# Introduction

The science of statistics deals with the collection, analysis, interpretation, and presentation of data. We see and use data in our everyday lives. In this session, you will learn how to organize and summarize data. Organizing and summarizing data is called descriptive statistics. Two ways to summarize data are by graphing and by using numbers (for example, finding an average). After you have studied probability and probability distributions, you will use formal methods for drawing conclusions from "good" data. The formal methods are called inferential statistics. Statistical inference uses probability to determine how confident we can be that our conclusions are correct.

In statistics, we generally want to study a population. You can think of a population as a collection of persons, things, or objects under study. To study the population, we select a sample. The idea of sampling is to select a portion (or subset) of the larger population and study that portion (the sample) to gain information about the population. Data are the result of sampling from a population.

Because it takes a lot of time and money to examine an entire population, sampling is a very practical technique. If you wished to compute the overall grade point average at your school, it would make sense to select a sample of students who attend the school. The data collected from the sample would be the students' grade point averages. In presidential elections, opinion poll samples of 1,000-2,000 people are taken. The opinion poll is supposed to represent the views of the people in the entire country. Manufacturers of canned carbonated drinks take samples to determine if a 16 ounce can contains 16 ounces of carbonated drink.

From the sample data, we can calculate a statistic. A statistic is a number that represents a property of the sample. For example, if we consider one math class to be a sample of the population of all math classes, then the average number of points earned by students in that one math class at the end of the term is an example of a statistic. The statistic is an estimate of a population parameter. A parameter is a numerical characteristic of the whole population that can be estimated by a statistic. Since we considered all math classes to be the population, then the average number of points earned per student over all the math classes is an example of a parameter.

One of the main concerns in the field of statistics is how accurately a statistic estimates a parameter. The accuracy really depends on how well the sample represents the population. The sample must contain the characteristics of the population in order to be a representative sample. We are interested in both the sample statistic and the population parameter in inferential statistics. In a later chapter, we will use the sample statistic to test the validity of the established population parameter.

A variable, usually notated by capital letters such as X and Y, is a characteristic or measurement that can be determined for each member of a population. Variables may be numerical or categorical. Numerical variables take on values with equal units such as weight in pounds and time in hours. Categorical variables place the person or thing into a category. If we let X equal the number of points earned by one math student at the end of a term, then X is a numerical variable. If we let Y be a person's party affiliation, then some examples of Y include Republican, Democrat, and Independent. Y is a categorical variable. We could do some math with values of X (calculate the average number of points earned, for example), but it makes no sense to do math with values of Y (calculating an average party affiliation makes no sense).

Data are the actual values of the variable. They may be numbers or they may be words. Datum is a single value.

## 14.1 Measures of Central Tendency

The "**center**" of a data set is also a way of describing location. The two most widely used measures of the "center" of the data are the **mean** (average) and the **median**. To calculate the mean weight of 50 people, add the 50 weights together and divide by 50. To find the median weight of the 50 people, order the data and find the number that splits the data into two equal parts. The median is generally a better measure of the center when there are extreme values or outliers because it is not affected by the precise numerical values of the outliers. The mean is the most common measure of the center. Another measure of the center is **mode**. It is the most frequently occurring data in a data set.

### Mean

Mean is obtained by dividing the sum of all observations in a data set by the total number of observations.

When the data is given with frequencies, the mean can be calculated by multiplying each distinct value by its frequency and then dividing the sum by the total number of data values. The letter used to represent the mean is an $x$ with a bar over it (pronounced "x bar"): $\bar{x}$

We use the following formula for finding the mean

$$\bar{x} = \frac{\text{sum of obserations}}{\text{total number of observations}}$$

For a grouped data,

$$\bar{x} = \frac{\sum fm}{\sum f}$$

where $f$ = the frequency of the interval and $m$ = the midpoint of the interval.

$$\text{Midpoint} = \frac{\text{lower boundary} \ + \ \text{upper boundary}}{2}$$

**Example 1**

AIDS data indicating the number of months a patient with AIDS lives after taking a new antibody drug are as follows (smallest to largest): 8, 10, 11, 11, 16, 17, 17, 24, 33, 33, 34, 44, and 47, then calculate the mean.

**Solution :**

The mean is

$$\bar{x} = \frac{8 + 10 + 11(2) + 16 + 17(2) + 24 + 33(2) + 34 + 44 + 47}{13} = 23.46$$

When each value in the data set is not unique, the mean can be calculated by multiplying each distinct value by its frequency and then dividing the sum by the total number of data values.

**Example 2**

In a small town of 50 people, one person earns $\$5,000$ per month and the other 49 each earn $\$3,000$. Find the mean?

**Solution :**

The mean is

$$\bar{x} = \frac{5000 + 49 \times 3000}{50} = 3040$$

**Example 3**

A frequency table displaying professor Blount's last statistic test is shown. Find the best estimate of the class mean.

| Grade interval | $50.5 - 56.5$ | $56.5 - 62.5$ | $62.5 - 68.5$ | $68.5 - 74.5$ | $74.5 - 80.5$ |
|---|---|---|---|---|---|
| Number of students | 1 | 0 | 4 | 4 | 2 |

**Solution :**

| Grade interval | Number of students ($f$) | Mid point value ($m$) | $fm$ |
|---|---|---|---|
| $50.5 - 56.5$ | 1 | 53.5 | 53.5 |
| $56.5 - 62.5$ | 0 | 59.5 | 0 |
| $62.5 - 68.5$ | 4 | 65.5 | 262 |
| $68.5 - 74.5$ | 4 | 71.5 | 286 |
| $74.5 - 80.5$ | 2 | 77.5 | 155 |

The mean is

$$\bar{x} = \frac{\sum fm}{\sum f} = \frac{53.5 + 0 + 262 + 286 + 155}{11} = 68.77$$

**Median**

You can quickly find the location of the median by using the expression $\frac{n+1}{2}$. The letter $n$ is the total number of data values in the sample.

If $n$ is an odd number, the median is the middle value of the ordered data (ordered smallest to largest).

If $n$ is an even number, the median is equal to the two middle values added together and divided by two after the data has been ordered. Remember that location of the median and the value of the median are not the same.

**Example 4**

Find the Median form the following data.

$$1, 11.5, 6, 7.2, 4, 8, 9, 10, 6.8, 8.3, 2, 2, 10, 1, 10$$

**Solution :**

Ordered from smallest to largest: $1, 1, 2, 2, 4, 6, 6.8, 7.2, 8, 8.3, 9, 10, 10, 11.5$

Since there are 14 observations, the median is between the seventh value, 6.8, and the eighth value, 7.2. To find the median, add the two values together and divide by two.

$$\frac{6.8 + 7.2}{2} = 7$$

So the median is 7.

**Example 5**

Find the Median for the data 2, 7, 4, 1, 3, 5, 8, 2, 4, 9, 1

**Solution :**

Order from smaller to larger: 1, 1, 2, 2, 3, 4, 4, 5, 7, 8, 9
Since there are 11 observations, the median is $\frac{11+1}{2} = 6$ th term. So the median is 4.

### Mode

The mode is the most frequent value. Number of times repeated and occurs in a given data.

- Unimodal means only 1 mode
- Bimodal means 2 modes
- Multimodal means more than 2 modes.

**Example 6**

Statistics exam scores for 20 students are as follows:

$$50, 53, 59, 59, 63, 63, 72, 72, 72, 72, 72, 76, 78, 81, 83, 84, 84, 84, 90, 93$$

Find the mode.

**Solution :**

The most frequent score is 72, which occurs five times. Mode = 72. [Unimodal]

**Example 7**

IELTS exam bandwidth score for 10 students are as follows:
$8, 3.5, 7, 6, 7, 6.5, 7.5, 4, 8, 7.2$
Find mode.

**Solution :**

The most frequent scores are 8 and 7, which occur 2 times [Bimodal]

## 14.1 EXERCISES

I. Answer the following questions:

(1) The data represent the number of days off per year for a sample of individuals selected from nine different countries. Find the mean.

$$20, 26, 40, 36, 23, 42, 35, 24, 30$$

(2) The number of guest-rooms of ten hotels in Muscat is $71, 30, 61, 59, 31, 41, 47, 64, 23$ and 22. Find the median.

(3) Find the median: $684, 764, 656, 702, 856, 1133, 1132$

(4) The number of rainy days for the top 10 rainy cities in last 3 years is shown. Find the median.

$$223, 211, 209, 227, 213, 240, 210, 228, 212, 240$$

(5) Find the mode for the following data.

$$110, 731, 1031, 84, 20, 118, 1162, 1977, 103, 752$$

(6) The following data are the distances between 10 retail stores and a large distribution center. The distances are in miles. Find the mode and type of mode?

(a) 86, 87, 150, 87, 40, 48, 50, 87, 58, 40

(b) 68, 69, 66, 96, 96, 99, 40, 68, 40, 57

(c) 29, 40, 58, 67, 95, 29, 11, 21, 45, 15

(7) Twelve major earthquakes had Richter magnitudes shown here.

$$7.0, 6.2, 7.7, 8.0, 6.4, 6.2, 7.2, 5.4, 6.4, 6.5, 7.2, 5.4$$

Find (a) the **mean**, (b) the **median**, and (c) the **mode**.

(8) Thirty automobiles were tested for fuel efficiency, in miles per gallon (mpg). The following frequency distribution was obtained. Find the mean for the data.

| Class Boundaries | Frequency |
|---|---|
| $7.5 - 12.5$ | 3 |
| $12.5 - 17.5$ | 5 |
| $17.5 - 22.5$ | 15 |
| $22.5 - 27.5$ | 5 |
| $27.5 - 32.5$ | 2 |

(9) Find the arithmetic mean for the following distribution

| Marks | Frequency |
|---|---|
| 45 | 4 |
| 30 | 1 |
| 85 | 3 |
| 90 | 5 |
| 75 | 3 |

(10)Find the arithmetic mean for the following distribution

| Class Intervals | Frequency |
|---|---|
| $5-10$ | 3 |
| $10-15$ | 1 |
| $15-20$ | 5 |
| $20-25$ | 3 |
| $25-30$ | 8 |

(11)Find the value of $P$ if the mean of the following distribution is 7.5.

| $x$ : | 3 | 5 | 7 | 9 | 11 | 13 |
|---|---|---|---|---|---|---|
| $f$ : | 6 | 8 | 15 | $P$ | 8 | 4 |

## 14.2 Summarizing Data into Tables and Graphs

In statistics we use various tables and diagrams to represent data which will facilitate easy interpretation and analysis of data.

### Frequency Distributions

In our discussion we discuss mainly two types of frequency distributions to represent data in tabular form.

**Categorical Frequency Distribution**

This distribution is used when the data can be categorized into different groups or categories. Here the data may be numeric or non-numeric.

| Classification | Frequency |
|---|---|
| Bank | 5 |
| Utilities | 4 |
| Properties | 8 |
| IT | 5 |
| Commerce | 12 |
| Industries | 6 |
| Total | 40 |

**Grouped Frequency Distribution**

This distribution is used to represent numerical data in classes and intervals. See the following example:

| Share Price | Number of Share holders |
|---|---|
| $1 - 20$ | 1000 |
| $21 - 40$ | 850 |
| $41 - 60$ | 450 |
| $61 - 80$ | 300 |
| $81 - 100$ | 250 |
| $101 - 120$ | 150 |
| Total | 3000 |

***How To* :: Constructing a grouped frequency distribution**




Step 1. Form the classes/class intervals. Pick out the highest and the lowest values and find the range of the data. Determine the class intervals. Number of intervals should be between 5 and 12 and they usually have equal widths. Ensure that each item of the data will be included in a unique class.

Step 2. Tally the values in the data set into the classes formed.

Step 3. Find the frequency of each class by totalling the tallies.

**Example 1**

Seventeen students were asked how many hours they studied per day. Their responses, in hours, are as follows:

$$5, 6, 3, 3, 4, 7, 5, 3, 5, 6, 5, 4, 4, 3, 5, 5, 3$$

Construct the frequency distribution table.

**Solution :**

| Table value | Frequency |
|---|---|
| 3 | 5 |
| 4 | 3 |
| 5 | 6 |
| 6 | 2 |
| 7 | 1 |

A relative frequency is the ratio (fraction or proportion) of the number of times a value of the data occurs in the set of all outcomes to the total number of outcomes. To find the relative frequencies, divide each frequency by the total number of students in the sample in this case, 20 . Relative frequencies can be written as fractions, percent's, or decimals.

Relative frequency $= \frac{f}{n} \times 100$ or Relative frequency $= \frac{f}{n}$

where $f$ is a frequency, $n$ is total number of frequency

Note: We can express relative error in 100% or decimal to 1

Cumulative relative frequency is the accumulation of the previous relative frequencies. To find the cumulative relative frequencies, add all the previous relative frequencies to the relative frequency for the current row.

**Example 2**

Twenty students were asked how many hours they worked per day. Their responses, in hours, are as follows.

$$5, 6, 3, 3, 2, 4, 7, 5, 2, 3, 5, 6, 5, 4, 4, 3, 5, 2, 5, 3$$

Construct frequency distribution

**Solution :**

| Data value | Tally | Frequency | Relative frequency in % | Cumulative relative frequency |
|---|---|---|---|---|
| 2 | /// | 3 | 15 | 15 |
| 3 | 𝍸 | 5 | 25 | $15 + 25 = 40$ |
| 4 | /// | 3 | 15 | $40 + 15 = 55$ |
| 5 | 𝍸/ | 6 | 30 | $55 + 30 = 85$ |
| 6 | // | 2 | 10 | $85 + 10 = 95$ |
| 7 | / | 1 | 5 | $95 + 5 = 100$ |

Relative frequency $= \frac{f}{n} \times 100$. The last entry of the cumulative relative frequency column is hundred, indicating that one hundred percent of the data has been accumulated.

Because of rounding, the relative frequency column may not always sum to hundred, and the last entry in the cumulative relative frequency column may not be hundred one. However, they each should be close to hundred.

**Example 3**

Grouped data: The table Represents the heights, in inches, of a sample of 100 males semiprofessional soccer players. Find relative frequency and cumulative relative frequency for the frequency distribution

| Heights (Inches) | 59.95 – 61.95 | 61.95 – 63.95 | 63.95 – 65.95 | 65.95 – 67.95 |
|---|---|---|---|---|
| Frequency | 5 | 3 | 15 | 40 |
| Heights (Inches) | 67.95 – 69.95 | 69.95 – 71.95 | 71.95 – 73.95 | 73.95 – 75.95 |
| Frequency | 17 | 12 | 7 | 1 |

**Solution :**

| Heights (Inches) | Tally | Frequency | $RF = \frac{f}{n}$ | Cumulative $RF$ |
|---|---|---|---|---|
| 59.95 – 61.95 | 𝍸 | 5 | 0.05 | 0.05 |
| 61.95 – 63.95 | /// | 3 | 0.03 | 0.05 + 0.03 = 0.08 |
| 63.95 – 65.95 | 𝍸𝍸 | 15 | 0.15 | 0.08 + 0.15 = 0.23 |
| 65.95 – 67.95 | 𝍸𝍸𝍸𝍸𝍸𝍸𝍸𝍸 | 40 | 0.4 | 0.23 + 0.40 = 0.63 |
| 67.95 – 69.95 | 𝍸𝍸𝍸// | 17 | 0.17 | 0.63 + 0.17 = 0.80 |
| 69.95 – 71.95 | 𝍸𝍸// | 12 | 0.12 | 0.80 + 0.12 = 0.92 |
| 71.95 – 73.95 | 𝍸// | 7 | 0.07 | 0.92 + 0.07 = 0.99 |
| 73.95 – 75.95 | / | 1 | 0.01 | 0.99 + 0.01 = 1 |
| Total | | 100 | | |

The last entry of the cumulative relative frequency column is one, indicating that one hundred percent of the data has been accumulated.

Because of rounding, the relative frequency column may not always sum to one, and the last entry in the cumulative relative frequency column may not be one. However, they each should be close to one.

### 14.2.1 Representing Data Using Graphs and Charts

- **Bar Charts**
  Bar charts (Bar Graphs/ Bar Diagrams) are used to represent categorical data. Bar graphs consist of bars that are separated from each other. The bars can be rectangles or they can be rectangular boxes (used in three-dimensional plots), and they can be vertical or horizontal.

- **Pie Charts**

Pie charts are also used to represent the categorical data. This representation gives emphasis to the relative weightage of each category. In a pie chart, a circle is drawn and it is divided into sectors. Number of sectors will be the number of categories. The area of each sector is proportional to the frequency of the categorical variable it represents.

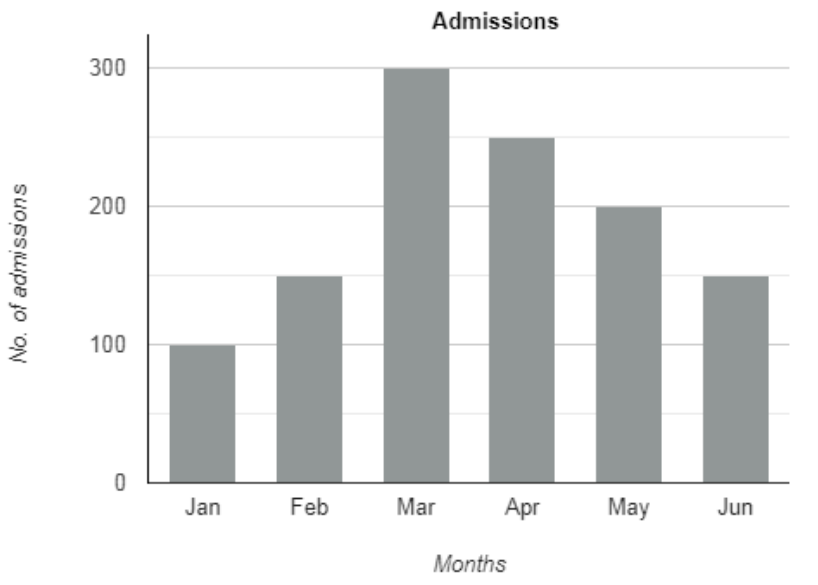
- **Histogram**
  A histogram consists of contiguous (adjoining) boxes. It has both a horizontal axis and a vertical axis. The horizontal axis is labeled with what the data represents (for instance, distance from your home to school). The vertical axis is labeled either frequency or relative frequency (or percent frequency or probability). The graph will have the same shape with either label. The histogram (like the stemplot) can give you the shape of the data, the center, and the spread of the data.

**Example 4**

The following table shows the number of admissions at a training institute for the months January to June. Construct a bar chart for the data.

| Month | Jan | Feb | Mar | Apr | May | Jun |
|---|---|---|---|---|---|---|
| No. of admissions | 100 | 150 | 300 | 250 | 200 | 150 |

**Solution :**



**Example 5**

In a college there are 400 students in the Foundation Program, 450 in the First Year Diploma, 350 in the Second Year Diploma and 250 in the Higher Diploma. Construct a pie chart for the data.

**Solution :**

Total number of students $= 400 + 450 + 350 + 250 = 1450$

Angle corresponding to Foundation Program $= \frac{400}{1450} \times 360° = 99.31°$

Percentage $= \frac{400}{1450} \times 100 = 27.6\%$

Angle corresponding to First Year Diploma $= \frac{450}{1450} \times 360° = 111.72°$

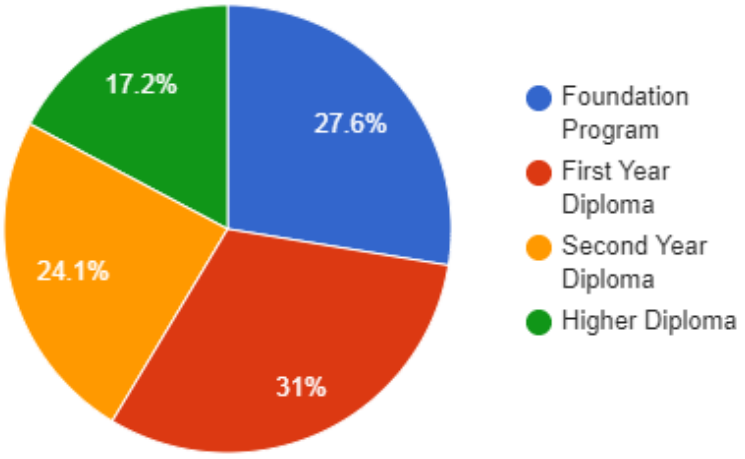
Percentage $= \frac{450}{1450} \times 100 = 31\%$

Angle corresponding to Second Year Diploma $= \frac{350}{1450} \times 360° = 86.90°$

Percentage $= \frac{350}{1450} \times 100 = 24.1\%$

Angle corresponding to Higher Diploma $= \frac{250}{1450} \times 360° = 62.07°$

Percentage $= \frac{250}{1450} \times 100 = 17.2\%$

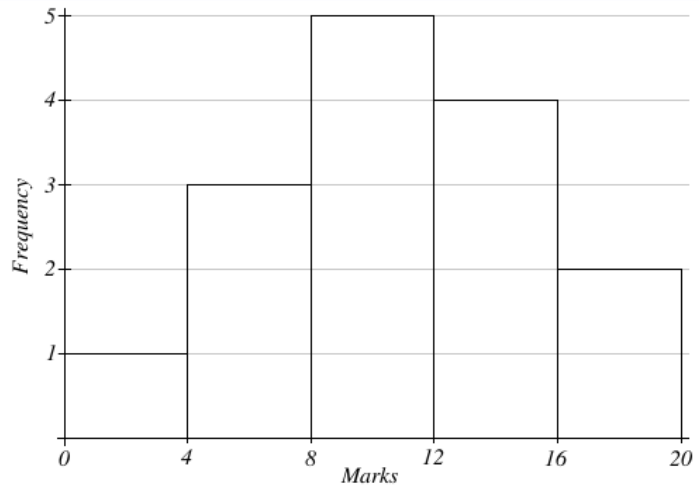


## Example 6

The following distribution shows the marks of 15 students of a class in the midterm examination. Construct a histogram for the data.

| Marks | 0-4 | 4-8 | 8-12 | 12-16 | 16-20 |
|---|---|---|---|---|---|
| No. of students | 1 | 3 | 5 | 4 | 2 |

## Solution :



## 14.2 EXERCISES

I. Answer the following questions:

(1) Thirty-five army cadets were given a blood test to determine their blood type. The data set is

| | | | | | | |
|---|---|---|---|---|---|---|
| A | O | B | B | B | AB | O |
| O | A | O | B | O | AB | B |
| B | O | B | O | A | A | O |
| A | O | O | O | O | O | AB |
| AB | A | AB | O | B | A | A |

Construct a frequency distribution for the data.

(2) A sample of 25 coloring pens is observed and their colors are recorded.

Green Blue Red Red Yellow Orange Red Blue Brown
Yellow Red Red Green Orange Yellow Blue Red Green
Red Brown Brown Yellow Blue Red Orange

Construct a frequency distribution

(3) The heights in inches of commonly grown herbs are given below. Organize the data into a frequency distribution with six classes.

15 36 18 20 18 18 20 18 18 24 20 10 24 16 16 20 7

(4) The average quantitative GRE scores for the top 30 graduate schools of engineering are listed. Construct a grouped frequency distribution with 5 classes.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 767 | 770 | 761 | 760 | 771 | 768 | 776 | 771 | 756 | 770 |
| 763 | 760 | 747 | 766 | 754 | 771 | 771 | 778 | 766 | 762 |
| 780 | 750 | 746 | 764 | 769 | 759 | 757 | 753 | 758 | 746 |

(5) The ages of the signers of the Declaration of Independence are shown. (Age is approximate since only the birth year appeared in the source, and one has been omitted since his birth year is unknown). Construct a grouped frequency distribution for the data using 7 classes.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 41 | 54 | 47 | 40 | 39 | 35 | 50 | 37 | 49 | 42 | 70 | 32 |
| 44 | 52 | 39 | 50 | 40 | 30 | 34 | 69 | 39 | 45 | 33 | 42 |
| 44 | 63 | 60 | 27 | 42 | 34 | 50 | 42 | 52 | 38 | 36 | 45 |
| 35 | 43 | 48 | 46 | 31 | 27 | 55 | 63 | 46 | 33 | 60 | 62 |
| 35 | 46 | 45 | 34 | 53 | 50 | 50 | | | | | |

(6) The table shows the performance of students in a class for the midterm exam

| Grade | A | B | C | D |
|---|---|---|---|---|
| No. of Students | 6 | 12 | 11 | 3 |

Represent the data using a **(i)** bar graph and **(ii)** a pie graph

(7) There are 216 employees in a factory as per list given below:

| Section | Electrician | Plumber | Fitter | Supervisor | Clerk |
|---|---|---|---|---|---|
| No. of Workers | 72 | 62 | 37 | 28 | 17 |

Represent the data using a **(i)** bar graph and **(ii)** a pie graph

(8) Construct a histogram to represent the data shown for the Blood Sugar levels of 60 inmates of an old age home.

| Class Boundaries | Frequency |
|---|---|
| $99.5 - 104.5$ | 2 |
| $104.5 - 109.5$ | 9 |
| $109.5 - 114.5$ | 20 |
| $114.5 - 119.5$ | 15 |
| $119.5 - 124.5$ | 8 |
| $124.5 - 129.5$ | 4 |
| $129.5 - 134.5$ | 2 |

(9) Thirty automobiles were tested for fuel efficiency, in miles per gallon (mpg). The following frequency distribution was obtained. Construct a histogram for the data.

| Class Boundaries | Frequency |
|---|---|
| $7.5 - 12.5$ | 3 |
| $12.5 - 17.5$ | 5 |
| $17.5 - 22.5$ | 15 |
| $22.5 - 27.5$ | 5 |
| $27.5 - 32.5$ | 2 |

(10) For 108 randomly selected college applicants, the following frequency distribution for entrance exam scores was obtained. Construct a histogram for the data.

| Class limits | Frequency |
|---|---|
| $90 - 98$ | 6 |
| $99 - 107$ | 22 |
| $108 - 116$ | 43 |
| $117 - 125$ | 28 |
| $126 - 134$ | 9 |

# Chapter 15

# Probability

**Contents**


**15.1 Basic Concepts of Probability**

**15.2 Probability Using Tree Diagrams**

15.2.1 Tree Diagrams using Frequencies

15.2.2 Tree Diagrams using Probabilities

**15.3 Permutations and Combinations**

**15.4 Probability Using Permutations and Combinations**


**Learning outcome covered:**

(z) `Understand basic probability concepts and compute the probability of simple events using tree diagrams and formulas for permutations and combinations.`

**Learning Objectives**

**By the end of this chapter, the students will be able to:**

- Understand the basic concepts of mathematical probability.
- Compute the probability of simple events.
- Compute probability using tree diagrams and permutations and combinations.

## Introduction:

Probability is a measure that is associated with how certain we are of outcomes of a particular experiment or activity. An experiment is a planned operation carried out under controlled conditions. If the result is not predetermined, then the experiment is said to be a **chance experiment**. Flipping one fair coin twice is an example of an experiment.

A result of an experiment is called an outcome. The sample space of an experiment is the set of all possible outcomes. Three ways to represent a sample space are: to list the possible outcomes, to create a tree diagram, or to create a Venn diagram. The uppercase letter $S$ is

used to denote the sample space. For example, if you flip one fair coin, $S = \{H, T\}$ where $H$ = heads and $T$ = tails are the outcomes.

An event is any combination of outcomes. Upper case letters like $A$ and $B$ represent events. For example, if the experiment is to flip one fair coin, event $A$ might be getting at most one head. The probability of an event $A$ is written $P(A)$.

## 15.1 Basic Concepts of Probability

The **probability** of any outcome is the long-term relative frequency of that outcome. Probabilities are between zero and one, inclusive (that is, zero and one and all numbers between these values). $P(A) = 0$ means the event $A$ can never happen. $P(A) = 1$ means the event $A$ always happens. $P(A) = 0.5$ means the event $A$ is equally likely to occur or not to occur.

The empty set $\phi$ and the sample space $S$ describe events. In fact $\phi$ is called an **impossible event** i.e. under no outcome of the experiment can this event be realized. A **null event** is an event that is impossible. Events that are impossible have zero probability, but the converse is not necessarily true. Clearly an empty set has zero probability. But, a zero probability event does not always mean an impossible event. For example, getting a number 9 while tossing a die is an impossible event as the only possible outcomes is between 1 to 6. The whole sample space $S$ is called a **sure event** which is the one that is defined to happen for sure. For example, getting a head (H) or tail (T) while tossing a coin once is a sure event as the outcome will definitely be either head or tail

**Equally likely** means that each outcome of an experiment occurs with equal probability. For example, if you toss a fair, six-sided die, each face $(1, 2, 3, 4, 5,$ or $6)$ is as likely to occur as any other face. If you toss a fair coin, a Head (H) and a Tail (T) are equally likely to occur. If you randomly guess the answer to a true/false question on an exam, you are equally likely to select a correct answer or an incorrect answer.

To calculate the probability of an event A when all outcomes in the sample space are equally likely, count the number of outcomes for event A and divide by the total number of outcomes in the sample space. For example, if you toss a fair dime and a fair nickel, the sample space is {HH,TH,HT,TT} where $T$ = tails and $H$ = heads. The sample space has four outcomes. $A$ = getting one head. There are two outcomes that meet this condition {HT,TH}, so $P(A) = \frac{2}{4} = 0.5$.

Suppose you roll one fair six-sided die, with the numbers $\{1, 2, 3, 4, 5, 6\}$ on its faces. Let event $E$ = rolling a number that is at least five. There are two outcomes $\{5, 6\}$. $P(E) = \frac{2}{6} = \frac{1}{3}$. If

you were to roll the die only a few times, you would not be surprised if your observed results did not match the probability. If you were to roll the die a very large number of times, you would expect that, overall, $\frac{2}{6}$ of the rolls would result in an outcome of "at least five". You would not expect exactly $\frac{2}{6}$. The long-term relative frequency of obtaining this result would approach the theoretical probability of $\frac{2}{6}$ as the number of repetitions grows larger and larger.

**"OR" Event**: An outcome is in the event $A$ OR $B$ if the outcome is in $A$ or is in $B$ or is in both $A$ and $B$. For example, let $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 5, 6, 7, 8\}$. $A$ OR $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Notice that 4 and 5 are NOT listed twice.

**"AND" Event**: An outcome is in the event $A$ AND $B$ if the outcome is in both $A$ and $B$ at the same time. For example let $A$ and $B$ be {1,2,3,4,5} and {4,5,6,7,8} , respectively. Then $A$ AND $B = \{4, 5\}$.

The **complement** of event $A$ is denoted $A'$(read as "$A$ prime"). $A'$ consists of all outcomes that are **NOT** in $A$. Notice that $P(A) + P(A') = 1$.

For example, let $S = \{1, 2, 3, 4, 5, 6\}$ and let $A = \{1, 2, 3, 4\}$.

Then, $A' = \{5, 6\}$. $P(A) = \frac{2}{6}$, $P(A') = \frac{4}{6}$, and

$$P(A) + P(A') = \frac{2}{6} + \frac{4}{6} = 1$$

The **conditional probability** of $A$ given $B$ is written $P(A/B)$. $P(A/B)$ is the probability that event $A$ will occur given that the event $B$ has already occurred. A conditional reduces the sample space. We calculate the probability of $A$ from the reduced sample space $B$. The formula to calculate $P(A/B)$ is

$$P(A/B) = \frac{P(A \text{ and } B)}{P(B)}$$

where $P(B)$ is greater than zero.

For example, suppose we toss one fair, six-sided die. The sample space $S = \{1, 2, 3, 4, 5, 6\}$. Let $A$ = face is 2 or 3 and $B$ = face is even $(2, 4, 6)$. To calculate $P(A/B)$, we count the number of outcomes 2 or 3 in the sample space $B = \{2, 4, 6\}$. Then we divide that by the number of outcomes $B$ (rather than $S$).

We get the same result by using the formula. Remember that $S$ has six outcomes.

$$P(A/B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{\left\{\frac{\text{the number of outcomes that are 2 or 3 and even in S}}{6}\right\}}{\left\{\frac{\text{the number of outcomes that are even in S}}{6}\right\}}$$

**Mutually Exclusive Events**: A and B are mutually exclusive events if they cannot occur at the same time. This means that $A$ and $B$ do not share any outcomes and $P(A \text{ AND } B) = 0$

For example, suppose the sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$, and $C = \{7, 9\}$. $A$ AND $B = \{4, 5\}$. $P(A \text{ AND } B) = \frac{2}{10}$ and is not equal to zero. Therefore, $A$ and $B$ are not mutually exclusive. $A$ and $C$ do not have any numbers in common so $P(A \text{ AND } C) = 0$. Therefore, $A$ and $C$ are mutually exclusive.

For example: Flip two fair coins (This is an experiment).

The sample space is $\{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}$ where T = tails and H = heads. The outcomes are HH, HT, TH, and TT. The outcomes HT and TH are different. The HT means that the first coin showed heads and the second coin showed tails. The TH means that the first coin showed tails and the second coin showed heads.

Let $A$ = the event of getting at most one tail. At most one tail means zero or one tail. Then A can be written as $\{\text{HH}, \text{HT}, \text{TH}\}$. The outcome HH shows zero tails. HT and TH each show one tail.

Let B = the event of getting all tails. B can be written as $\{\text{TT}\}$. B is the complement of A, so $\text{B} = \text{A}'$. Also, $\text{P(A)} + \text{P(B)} = \text{P(A)} + \text{P}(\text{A}') = 1$.

The probabilities for A and for B are $\text{P(A)} = \frac{3}{4}$ and $\text{P(B)} = \frac{1}{4}$

Let C = the event of getting all heads. $\text{C} = \{\text{HH}\}$. since $\text{B} = \{\text{TT}\}$, P(B AND C) = 0. Thus B and C are mutually exclusive. $B$ and $C$ have no members in common because you cannot have all tails and all heads at the same time.

Let $D$ = event of getting more than one tail. $D = \{\text{TT}\}$. $\text{P(D)} = \frac{1}{4}$

Let E = event of getting a head on the first roll. This implies you can get either a head or tail on the second roll. $\text{E} = \{\text{HT}, \text{HH}\}$ and $\text{P(E)} = \frac{2}{4}$.

Let F = event of getting at least one (one or two) tail in two flips. $\text{F} = \{\text{TT}, \text{HT}, \text{TH}\}$ and

$$P(F) = \frac{3}{4}.$$

### Complement Rule to Compute Probabilities

We have discussed how to calculate the probability that an event will happen. Sometimes, we are interested in finding the probability that an event will not happen. The complement of an event E, denoted E′, is the set of outcomes in the sample space that are not in E.

For example, suppose we are interested in the probability that a horse will lose a race. If event W is the horse winning the race, then the complement of event W′ is the horse losing the race.

To find the probability that the horse loses the race, we need to use the fact that the sum of all probabilities in a probability model must be 1.

$$P(W') = 1 - P(W)$$

The probability of the horse winning added to the probability of the horse losing must be equal to 1. Therefore, if the probability of the horse winning the race is $\frac{1}{9}$, the probability of the horse losing the race is simply

$$1 - \frac{1}{9} = \frac{8}{9}$$

**Example 1**

If the probability of getting rain tomorrow is 0.35, find the probability that it will not rain tomorrow.

**Solution :**

If the event E is getting rain tomorrow, then the probability of getting rain tomorrow is P(E)= 0.35, To find the probability that it will not rain tomorrow is

$$P(E') = 1 - P(E)$$
$$P(E') = 1 - 0.35 = 0.65$$

**Example 2**

What is the probability that when two dice are thrown, the number on each die will not be the same?

**Solution :**

This is the same as saying that doubles will not occur. Since the probability of doubles

occurring is $\frac{6}{36} = \frac{1}{6}$, then the probability that doubles will NOT occur is:

$$1 - \frac{1}{6} = \frac{6}{6} - \frac{1}{6} = \frac{5}{6}$$

**Two Basic Rules of Probability**

**The Multiplication Rule**

If $A$ and $B$ are two events defined on a sample space, then: $P(A \text{ AND } B) = P(B)P(A/B)$

This rule may also be written as:

$$P(A/B) = \frac{P(A \text{ AND } B)}{P(B)}$$

If A and B are independent, then P(A/B) = P(A).

Then $P(A \text{ AND } B) = \text{P(A/B)P(B)}$ becomes $P(A \text{ AND } B) = \text{P(A)P(B)}$

**The Addition Rule**

If $A$ and $B$ are defined on a sample space, then:

$$P(A \text{ OR } B) = P(A) + P(B) - P(A \text{ AND } B)$$

If A and B are mutually exclusive, then $P(A \text{ AND } B) = 0$

Then $P(A \text{ OR } B) = \text{P(A)} + \text{P(B)} - \text{P(A AND B)}$ becomes $P(A \text{ OR } B) = \text{P(A)} + \text{P(B)}$.

**Example 3**

If a die is rolled,

(a) what is the probability of obtaining an even number or a number greater than four?

(b) what is the probability of obtaining an even number and a number greater than four?

**Solution :**

Let $E$ be the event that the number shown on the die is an even number, and let $F$ be the event that the number shown is greater than four. The sample space $S = \{1, 2, 3, 4, 5, 6\}$ The event $E = \{2, 4, 6\}$, and the event $F = \{5, 6\}$

(a) Even number or Greater than 4
$E \cup F = \{2, 4, 5, 6\}$

Therefore, $P(E \cup F) = \frac{4}{6}$

(b) Even number and Greater than 4

$E \cap F = \{6\}$

Therefore, $P(E \cap F) = \frac{1}{6}$

**Example 4**

Two dice are rolled, and the events F and T are as follows:
$F =$ { The sum of the dice is 4} and $T =$ { At least one die shows a three } Find $P(F \cup T)$.

**Solution :**

We list F and T, and $F \cap T$ as follows:

$F = \{(1,3),(2,2),(3,1)\}$

$T = \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(1,3),(2,3),(4,3),(5,3),(6,3)\}$

$F \cap T = \{(1,3),(3,1)\}$

Since $P(F \cup T) = P(F) + P(T) - P(F \cap T)$

We have, $P(F \cup T) = \frac{3}{36} + \frac{11}{36} - \frac{2}{36} = \frac{12}{36}$

**Example 5**

Klaus is trying to choose where to go on vacation. His two choices are: A = New Zealand and B = Alaska. Klaus can only afford one vacation. The probability that he chooses A is P(A) = 0.6 and the probability that he chooses B is P(B) = 0.35. Find the probability that he does not choose to go anywhere on vacation.

**Solution :**

P(A AND B) = 0 because Klaus can only afford to take one vacation.

Therefore, the probability that he chooses either New Zealand or Alaska is

$$P(A \text{ OR } B) = P(A) + P(B) = 0.6 + 0.35 = 0.95$$

Note that the probability that he does not choose to go anywhere on vacation must be 0.05.

**Example 6**

Carlos plays college soccer. He makes a goal 65 % of the time he shoots. Carlos is going to attempt two goals in a row in the next game. A = the event Carlos is successful on his first attempt. $P(A) = 0.65$. B = the event Carlos is successful on his second attempt. $P(B) = 0.65$. Carlos tends to shoot in streaks. The probability that he makes the second goal given that he made the first goal is 0.90.

(a) What is the probability that he makes both goals?

(b) What is the probability that Carlos makes either the first goal or the second goal?

(c) Are A and B independent?

(d) Are A and B mutually exclusive?

**Solution :**

(a) The problem is asking you to find $P(A \text{ AND } B) = P(B \text{ AND } A)$.
Since $P(B/A) = 0.90$, $P(B \text{ AND } A) = P(B/A)P(A) = 0.90 \times 0.65 = 0.585$
$\therefore$ Carlos makes the first and second goals with probability 0.585

(b) The problem is asking you to find $P(A \text{ OR } B)$
$P(A \text{ OR } B) = P(A) + P(B) - P(A \text{ AND } B) = 0.65 + 0.65 - 0.585 = 0.715$
$\therefore$ Carlos makes either the first goal or the second goal with probability 0.715

(c) No, they are not, because $P(B \text{ AND } A) = 0.585$
$P(B)P(A) = 0.65 \times 0.65 = 0.423$
$P(B \text{ AND } A) \neq P(B) \times P(A)$

(d) No, they are not because $P(A \text{ AND } B) = 0.585$
To be mutually exclusive, $P(A \text{ AND } B)$ must equal zero.

**Example 7**

A single die is rolled. Find the conditional probability of obtaining an even number given that a number greater than three has shown.

**Solution :**

Let $E$ be the event that an even number shows and F be the event that a number greater than three shows. We want $P(E \mid F)$.

$$E = \{2, 4, 6\} \text{ and } F = \{4, 5, 6\}$$

Which implies, $E \cap F = \{4, 6\}$

Therefore, $P(F) = \frac{3}{6}$, and $P(E \cap F) = \frac{2}{6}$. So

$$P(F) = \frac{P(E \cap F)}{P(F)} = \frac{2/6}{3/6} = \frac{2}{3}$$

**Example 8**

Given $P(E) = 0.5, P(F) = 0.7$, and $P(E \cap F) = 0.3$. Find

(a) $P(E \mid F)$

(b) $P(F \mid E)$

**Solution :**

We use the conditional probability formula $P(F) = \frac{P(E \cap F)}{P(F)}$

(a) $P(E \mid F) = \frac{0.3}{0.7} = \frac{3}{7}$

(b) $P(F \mid E) = \frac{0.3}{0.5} = \frac{3}{5}$

## 15.1 EXERCISES

I. Answer the following questions:

(1) Find the sample space for rolling two dice.

(2) Find the sample space for drawing one card from an ordinary deck of cards.

(3) Find the sample space for the gender of the children if a family has three children. Use B for boy and G for girl.

(4) A die is rolled, find the probability that an even number is obtained.

(5) Two coins are tossed, find the probability that two heads are obtained.

(6) Two dice are rolled, find the probability that the sum is

(a) equal to 1 (b) equal to 4 (c) less than 13

(7) A die is rolled and a coin is tossed, find the probability that the die shows an odd number and the coin shows a head.

(8) A card is drawn at random from a deck of cards. Find the probability of getting the 3 of diamonds.

(9) A card is drawn at random from a deck of cards. Find the probability of getting a queen.

(10)A jar contains 3 red marbles, 7 green marbles and 10 white marbles. If a marble is drawn from the jar at random, what is the probability that this marble is white?

(11)A die is rolled, find the probability that the number obtained is greater than 4.

(12)Two coins are tossed, find the probability that exactly one head is obtained.

(13)Two dice are rolled, find the probability that the sum is equal to 5.

(14)A card is drawn at random from a deck of cards. Find the probability of getting the King of hearts.

(15)Compute the probability of randomly drawing five cards from a deck and getting exactly one Ace.

# 15.2 Probability Using Tree Diagrams

**Tree Diagrams**

A tree diagram is a special type of graph used to determine the outcomes of an experiment. It consists of “branches" that are labeled with either frequencies or probabilities. Tree diagrams can make some probability problems easier to visualize and solve. The following example illustrates how to use a tree diagram.

## 15.2.1 Tree Diagrams using Frequencies

**Example 1**

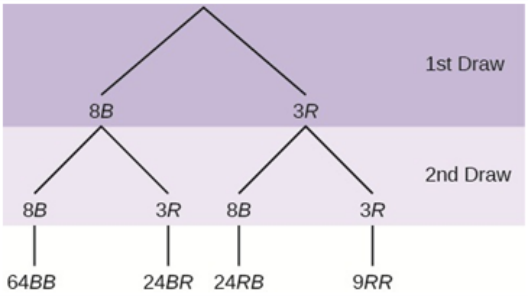
In an urn, there are 11 balls. Three balls are red (R) and eight balls are blue (B). Draw two balls, one at a time, with replacement, from the urn. Find the probability to draw both R.

**Solution :**

“With replacement" means that you put the first ball back before you select the second ball.

The tree diagram using frequencies that show all the possible outcomes follows.

$$\text{Total} = 64 + 24 + 24 + 9 = 121$$




The first set of branches represents the first draw. The second set of branches represents the second draw. Each of the outcomes is distinct.

In fact, we can list each red ball as R1, R2, and R3 and each blue ball as B1, B2, B3, B4, B5, B6, B7, and B8.

Then the nine RR outcomes can be written as: R1R1; R1R2; R1R3; R2R1; R2R2; R2R3; R3R1; R3R2; R3R3

The other outcomes are similar. There are a total of 11 balls in the urn.

Draw two balls, one at a time, with replacement. There are $11 \times 11 = 121$ outcomes, the size of the sample space.

### 15.2.2 Tree Diagrams using Probabilities

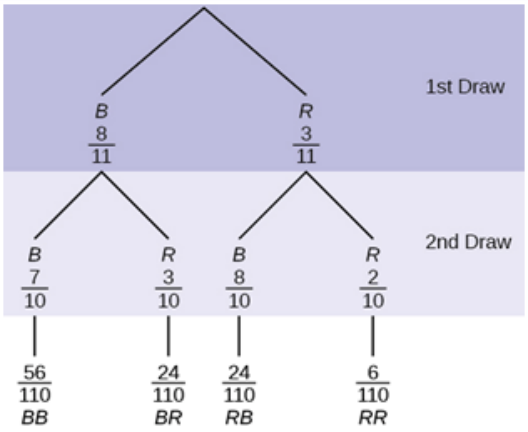
**Example 2**

An urn has three red marbles and eight blue marbles in it. Draw two marbles, one at a time, this time without replacement, from the urn. Find the probability to draw both R.

**Solution :**

"Without replacement" means that you do not put the first ball back before you select the second marble.

Following is a tree diagram for this situation. The branches are labeled with probabilities instead of frequencies. The numbers at the ends of the branches are calculated by multiplying the numbers on the two corresponding branches, so the probability to draw both R is $\left(\frac{3}{11}\right)\left(\frac{2}{10}\right) = \frac{6}{110}$.




$$\text{Total} = \frac{56+24+24+6}{110} = \frac{110}{110} = 1$$

## 15.2 EXERCISES

I. Answer the following questions:

(1) Two coins are tossed together. Find the following probabilities using a tree diagram.

(a) Probability of getting two heads

(b) Probability of getting at least one head

(c) Probability of getting at most one head

(d) Probability of getting exactly one head

(2) Bag A contains three red beads and four blue beads. Bag B contains five red beads and three blue beads. A beads is taken from each bag in turn. Answer the following questions using a tree diagram.

   (a) What is the probability of getting a blue bead followed by a red?
   (b) What is the probability of getting a bead of each colour?

(3) Suppose there are five red balls and four yellow balls in a box. Two balls are drawn from the box without replacement one after the other. Using a tree diagram find the probability that one ball of each color is selected?

(4) A coin is biased so that the chance of getting a head is 60%. If it is tossed three times, compute the probability of getting the following events using a tree diagram.

   (a) three tails
   (b) exactly 2 heads
   (c) at least one tail

(5) Nasser has a bag with six red pencils and four blue pencils in it. He picks up a pencil from the bag at random, and then picks another at random after replacing the first pencil. With the help of a tree diagram calculate the probabilities that he picks:

   (a) two blue pencils
   (b) one pencil of each color
   (c) no blue pencils
   (d) at least one red pencil

(6) Nasser has a bag with eight red pencils and four blue pencils in it. He picks up a pencil from the bag at random, and then picks another at random without replacing the first pencil. With the help of a tree diagram calculate the probabilities that he picks:

   (a) two blue pencils
   (b) one pencil of each color
   (c) no blue pencils
   (d) at least one red pencil

## 15.3 Permutations and Combinations

### Permutations

The Multiplication Principle can be used to solve a variety of problem types. One type of problem involves placing objects in order. We arrange letters into words and digits into numbers, line up for photographs, decorate rooms, and more. An ordering of objects is called a **permutation**.

### Finding the Number of Permutations of $n$ Distinct Objects Using the Multiplication Principle

To solve permutation problems, it is often helpful to draw line segments for each option. That enables us to determine the number of each option so we can multiply.
For instance, suppose we have four paintings, and we want to find the number of ways we can hang three of the paintings in order on the wall. We can draw three lines to represent the three places on the wall.

$$\_\_\_ \times \_\_\_ \times \_\_\_$$

There are four options for the first place, so we write a 4 on the first line.

$$4 \times \_\_\_ \times \_\_\_$$

After the first place has been filled, there are three options for the second place so we write a 3 on the second line.

$$4 \times 3 \times \_\_\_$$

After the second place has been filled, there are two options for the third place so we write a 2 on the third line. Finally, we find the product.

$$4 \times 3 \times 2$$

There are 24 possible permutations of the paintings.

### Example 1

At a swimming competition, nine swimmers compete in a race. How many ways can they place first, second, and third?

**Solution :**

$$9 \times 8 \times 7 = 504$$

### Finding the Number of Permutations of n Distinct Objects Using Formula:

Given $n$ distinct objects, the number of ways to select $r$ objects from the set in order is

$$P(n,r) = \frac{n!}{(n-r)!}$$

**Example 2**

A professor is creating an exam of 9 questions from a test bank of 12 questions. How many ways can she select and arrange the questions?

**Solution :**

$$P(12,9) = \frac{12!}{(12-9)!} = 79,833,600$$

**Example 3**

A play has a cast of 7 actors preparing to make their curtain call. Use the permutation formula to find how many ways 5 of the 7 actors can be chosen to line up.

**Solution :**

$$P(7,5) = \frac{7!}{2!} = 2520$$

### Finding the Number of Permutations of $n$ Non-Distinct Objects

We have studied permutations where all of the objects involved were distinct. What happens if some of the objects are indistinguishable? For example, suppose there is a sheet of 12 stickers. If all of the stickers were distinct, there would be 12! ways to order the stickers. However, 4 of the stickers are identical stars, and 3 are identical moons. Because all of the objects are not distinct, many of the 12! permutations we counted are duplicates. The general formula for this situation is as follows.

$$\frac{n!}{r_1!r_2!\ldots r_k!}$$

In this example, we need to divide by the number of ways to order the 4 stars and the ways to order the 3 moons to find the number of unique permutations of the stickers. There are 4! ways to order the stars and 3! ways to order the moon.

$$\frac{12!}{4!3!} = 3,326,400$$

There are 3,326,400 ways to order the sheet of stickers.

### Formula for finding the number of permutations of $n$ non-distinct objects

If there are $n$ elements in a set and $r_1$ are alike, $r_2$ are alike, $r_3$ are alike, and so on through $r_k$, the number of permutations can be found by

$$\frac{n!}{r_1!r_2!\ldots r_k!}$$

**Example 4**

Finding the Number of Permutations of $n$ Non-Distinct Objects Find the number of rearrangements of the letters in the word "DISTINCT".

**Solution :**

There are 8 letters. Both I and T are repeated 2 times. Substitute $n = 8, r_1 = 2$, and $r_2 = 2$ into the formula.

$$\frac{8!}{2!2!} = 10,080$$

There are $10,080$ arrangements.

**Combinations**

So far, we have looked at problems asking us to put objects in order. There are many problems in which we want to select a few objects from a group of objects, but we do not care about the order. When we are selecting objects and the order does not matter, we are dealing with **combinations**. A selection of $r$ objects from a set of $n$ objects where the order does not matter can be written as $C(n, r)$.

**Find the Number of Combinations Using the Formula:**

Given $n$ distinct objects, the number of ways to select $r$ objects from the set is

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

**Example 5**

A fast food restaurant offers five side dish options. Your meal comes with two side dishes.

(a) How many ways can you select your side dishes?

(b) How many ways can you select 3 side dishes?

**Solution :**

(a)

$$C(5, 2) = \frac{5!}{2!(5-2)!} = 10$$

(b)

$$C(5,3) = \frac{5!}{3!(5-3)!} = 10$$

**Example 6**

How many line segments can be drawn by connecting any two of the six points that lie on the circumference of a circle?

**Solution :**

$$C(6,2) = 15$$

**Example 7**

There are ten people at a party. If they all shake hands, how many hand-shakes are possible?

**Solution :**

$$C(10,2) = 45$$

## 15.3 EXERCISES

I. Answer the following questions:

(1) Find the number of words, with or without meaning, that can be formed with the letters of the word

(a) OMAN
(b) CHAIR
(c) SWIMMING
(d) BUTTER

(2) How many 4 digit numbers that are divisible by 10 can be formed from the numbers 3,5 7,8,9,0 such that no number repeats?

(3) We need to form a 5 a side team in a class of 12 students. How many different teams can be formed?

(4) How many 4 digit numbers can we make using the digits 3,6,7 and 8 without repetitions?

(5) How many 3 digit numbers can we make using the digits 2, 3, 4, 5, and 6 without repetitions?

(6) How many 6 letter words can we make using the letters in the word 'LIBERTY' without repetitions?

(7) In how many ways can you arrange 5 different books on a shelf?

(8) In how many ways can you select a committee of 3 students out of 10 students?

(9) A family of five is having portraits taken. Use the Multiplication Principle to find how many ways the photographer can line up 3 of the family members.

(10) A restaurant offers a breakfast special that includes a breakfast sandwich, a side dish, and a beverage. There are 3 types of breakfast sandwiches, 4 side dish options, and 5 beverage choices. Find the total number of possible breakfast specials.

(11) How many ways are there to construct a string of 3 digits if numbers cannot be repeated?

(12) How many ways are there to construct a string of 3 digits if numbers can be repeated?

(13) How many ways are there to pick a red ace or a club from a standard card playing deck?

(14) How many ways are there to pick a paint color from 5 shades of green, 4 shades of blue, or 7 shades of yellow?

(15) A family consisting of 2 parents and 3 children is to pose for a picture with 2 family members in the front and 3 in the back.

(a) How many arrangements are possible with no restrictions?
(b) How many arrangements are possible if the parents must sit in the front?
(c) How many arrangements are possible if the parents must be next to each other?

(16)How many ways can a baseball coach arrange the order of 9 batters if there are 15 players on the team?

(17)How many ways can a committee of 3 freshmen and 4 juniors be formed from a group of 8 freshmen and 11 juniors?

(18)A skateboard shop stocks 10 types of board decks, 3 types of trucks, and 4 types of wheels. How many different skateboards can be constructed?

(19)Fatima bought 20 plants to arrange along the border of her garden. How many distinct arrangements can she make if the plants are comprised of 6 tulips, 6 roses, and 8 daisies?

(20)A wholesale T-shirt company offers sizes small, medium, large, and extra-large in organic or non-organic cotton and colors white, black, gray, blue, and red. How many different T-shirts are there to choose from?

(21)Salim wants to place billboard advertisements throughout the municipality for his new business. How many ways can he choose 15 neighborhoods to advertise in if there are 30 neighborhoods in the municipality?

(22)An art store has 4 brands of paint pens in 12 different colors and 3 types of ink. How many paint pens are there to choose from?

## 15.4 Probability Using Permutations and Combinations

Many interesting probability problems involve counting principles, permutations, and combinations. In these problems, we will use permutations and combinations to find the number of elements in events and sample spaces. These problems can be complicated, but they can be made easier by breaking them down into smaller counting problems.

**Example 1**

A 3−digit PIN is selected. What is the probability that there are no repeated digits?

**Solution :**

The digits of PIN can have any three numbers from 0,1,2,3,4,5,6,7,8,9 with the condition that no repeated selection. Here order of selection also taken into account. So we use permutation.

$$\text{Probablity} = \frac{\text{Permutation of selecting 3-digit PIN without repeated digits}}{\text{Permutation of all possibilities}}$$

The numerator in the equation is permutations without repetition and the denominator in the equation is permutations with repetition.

$$= \frac{P(10,3)}{10^3} = \frac{720}{1000} = 0.72$$

The probability of selecting 3 digit pin with no repeated digits is 0.72

**Example 2**

In a dance competition, the order of each performance is selected randomly. The dancers Ahmed, Anwar, Ziyah, Safa, and Abdullah are participating in it. What is the probability of Safa will go second?

**Solution :**

There are five performance in the competition with five dancers. Here order of selection also taken into account. So we use permutation.

$$\text{Probablity} = \frac{\text{Permutation of keeping Safa second}}{\text{Permutation of all possibilities}}$$

The numerator in the equation is permutations keeping Safa as second participant (if Safa is assigned for second, then rest of the 4 places will be randomly selected for four dancers) and the denominator in the equation is permutations of all possibilities with five place and five dancers.

$$= \frac{P(4,4)}{P(5,5)} = \frac{4!}{5!} = \frac{24}{120} = 0.2$$

The probability of Safa will go as second participant in a dance competition is 0.2

**Example 3**

A store has 8 cellular phones and that 3 of those are defective. Find the probability that a couple purchasing 2 phones receives 2 phones that are not defective.

**Solution :**

To solve this problem, we need to calculate all of the ways to select 2 phones that are not defective as well as all of the ways to select 2 phones.

There are 5 phones that are not defective, so there are C(5,2) ways to select 2 phones that are not defective.

There are 8 phones, so there are C(8,2) ways to select 2 phones. Thus the probability of selecting 2 phones that are not defective is:

$$\frac{\text{ways to select 2 phones that are not defective}}{\text{ways to select 2 phones}} = \frac{C(5,2)}{C(8,2)}$$

x

$$= \frac{10}{28} = \frac{5}{14}$$

**Example 4**

A child randomly selects 5 toys from a bin containing 3 bunnies, 5 dogs, and 6 bears. Find the probability that only bears are chosen.

**Solution :**

We need to count the number of ways to choose only bears and the total number of possible ways to select 5 toys.

There are 6 bears, so there are C(6, 5) ways to choose 5 bears. There are 14 toys, so there are C(14, 5) ways to choose any 5 toys.

$$\frac{C(6,5)}{C(14,5)} = \frac{6}{2002} = \frac{3}{1001}$$

**Example 5**

A child randomly selects 5 toys from a bin containing 3 bunnies, 5 dogs, and 6 bears. Find the probability that 2 bears and 3 dogs are chosen.

**Solution :**

We need to count the number of ways to choose 2 bears and 3 dogs and the total number of possible ways to select 5 toys.

There are 6 bears, so there are C(6, 2) ways to choose 2 bears. There are 5 dogs, so there are C(5, 3) ways to choose 3 dogs.

Since we are choosing both bears and dogs at the same time, we will use the Multiplication Principle.
There are C(6, 2) . C(5, 3) ways to choose 2 bears and 3 dogs.

We can use this result to find the probability.

$$\frac{C\,(6,2)\;C\,(5,3)}{C\,(14,5)} = \frac{15.10}{2002} = \frac{75}{1001}$$

**Example 6**

A child randomly selects 5 toys from a bin containing 3 bunnies, 5 dogs, and 6 bears. Find the probability that at least 2 dogs are chosen.

**Solution :**

It is often easiest to solve "at least" problems using the Complement Rule. We will begin by finding the probability that fewer than 2 dogs are chosen. If less than 2 dogs are chosen, then either no dogs could be chosen, or 1 dog could be chosen.

**Case 1 :** When no dogs are chosen (Event A), all 5 toys come from the 9 toys that are not dogs.

There are $C(9,5)$ ways to choose toys from the 9 toys that are not dogs.

Since there are 14 toys, there are $C(14,5)$ ways to choose the 5 toys from all of the toys.

$$P(A) = \frac{C\,(9,5)}{C\,(14,5)} = \frac{63}{1001}$$

**Case 2 :** When there is 1 dog chosen (Event B), then 4 toys must come from the 9 toys that are not dogs, and 1 must come from the 5 dogs. Since we are choosing both dogs and other toys at the same time, we will use the Multiplication Principle.

There are $C(5,1)$. $C(9,4)$ ways to choose 1 dog and 1 other toy.

$$\begin{aligned} P(B) &= \frac{C\,(5,1)\;C\,(9,4)}{C\,(14,5)} \\ &= \frac{5 \times 126}{2002} \\ &= \frac{315}{1001} \end{aligned}$$

Because these events would not occur together and are therefore mutually exclusive, we add the probabilities to find the probability that fewer than 2 dogs are chosen.

$$\begin{aligned} P(A \text{ OR } B) &= P(A) + P(B) \\ &= \frac{63}{1001} + \frac{315}{1001} \\ &= \frac{378}{1001} \end{aligned}$$

We then subtract that probability from 1 to find the probability that at least 2 dogs are chosen.

$$1 - P(A \text{ OR } B) = 1 - \frac{378}{1001} = \frac{623}{1001}$$

## 15.4 EXERCISES

I. Answer the following questions:

(1) A 4-digit PIN is selected. What is the probability that there are no repeated digits?

(2) The books namely Math, English, Arabic, Physics, and Chemistry are arranged in the rack by the librarian. What is the probability of Math book will be kept in first place?

(3) Ahmed belongs to a class of 20 students. In his class, 7 students are randomly selected to give the presentation. What is the probability of Ahmed got selected for the presentation?

(4) A bag contains 3 red and 5 black balls. Two balls are drawn at random from the bag. What is the probability that the balls drawn are both red?

(5) A bag contains 4 blue and 5 green balls. Two balls are drawn at random from the bag. What is the probability that the balls drawn are both green?

(6) Suppose you have a bag of 20 blue marbles and 40 red marbles. What is the probability that if 5 are drawn without replacement, 2 are blue and 3 are red?

(7) A child randomly selects 3 gumballs from a container holding 4 purple gumballs, 8 yellow gumballs, and 2 green gumballs.

(a) Find the probability that all 3 gumballs selected are purple.

(b) Find the probability that no yellow gumballs are selected.

(c) Find the probability that at least 1 yellow gumball is selected.

# Chapter 16

# Introduction of GeoGebra

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**Learning outcome covered:**

(x) `Use appropriate softwares to interpret equations and graphs.`

**Learning Objectives**

**By the end of this chapter, the students will be able to:**

- Sketch and interpret graphs of basic functions.
- Interpret equations and inequalities graphically.

## Introduction:

In this chapter, we will learn to sketch and interpret polynomials, linear and quadratic equations using dynamic software. GeoGebra is a dynamic mathematics software and an interactive geometry, algebra, statistics and calculus application, intended for learning and teaching mathematics and science from primary school to university level. It is available on multiple platforms with its desktop applications for Windows, macOS and Linux, with its tablet apps for Android, iPad and Windows, and with its web application based on HTML5 technology.

You can do constructions with points, vectors, segments, lines, polygons and conic sections as well as functions while changing them dynamically afterwards. On the other hand, equations and coordinates can be entered directly. Thus, GeoGebra has the ability to deal with variables for numbers, vectors and points. It finds derivatives and integrals of functions and offers commands like Root or Vertex.

## 16.1 GeoGebra

### 16.1.1 GeoGebra Installers

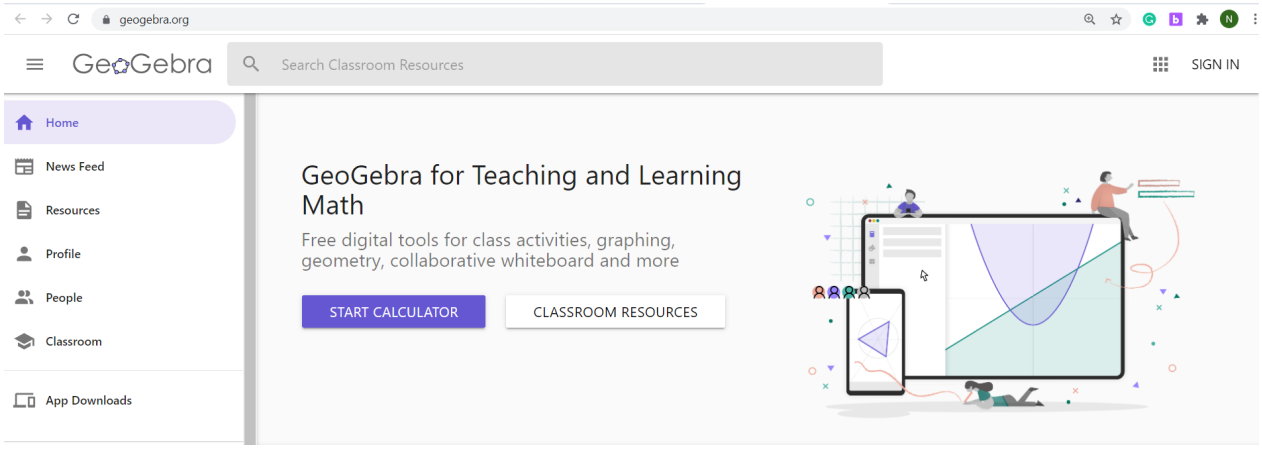
Download the installer file from http://www.geogebra.org/download

Make sure you have the correct version for your operating system.

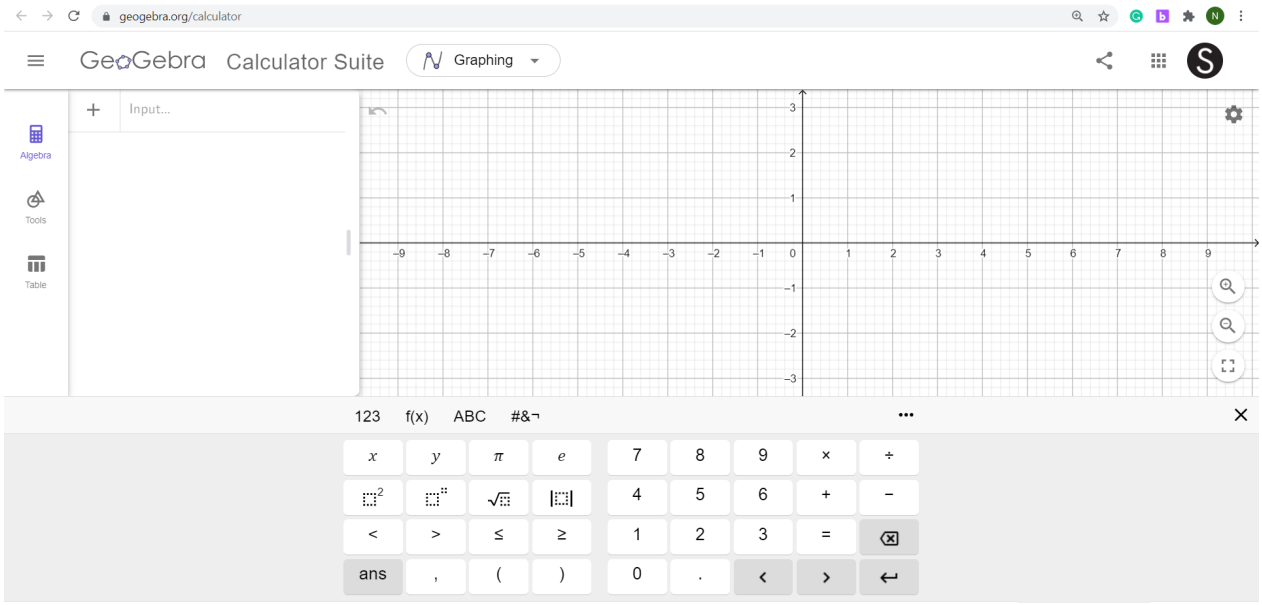
Double-click the GeoGebra installer file and follow the instructions of the installer assistant.

**Reach GeoGebra (without installation):**

- Step-1 : Use any browser and reach GeoGebra offical website : http://www.geogebra.org
- Step-2 : Click "START CALCULATOR"




- Step-3 : In the input bar, type the required function to generate its graph




### GeoGebra's User Interface

After starting GeoGebra, Using the provided geometry tools in the Toolbar you can create geometric constructions on the Graphics View with your mouse. At the same time the corresponding coordinates and equations are displayed in the Algebra View. On the other hand, you can directly enter algebraic input, commands, and functions into the Input Bar by using the keyboard. While the graphical representation of all objects is displayed in the Graphics View, their algebraic numeric representation is shown in the Algebra View. In GeoGebra, geometry and algebra work side by side.

### How to Operate GeoGebra's Geometry Tools

- Activate a tool by clicking on the button showing the corresponding icon.
- Open a toolbox by clicking on the lower part of a button and select another tool from this toolbox. Hint: You don't have to open the toolbox every time you want to select a tool. If the icon of the desired tool is already shown on the button it can be activated directly. Hint: Toolboxes contain similar tools or tools that generate the same type of new object.
- Click on the icon at the right of the Toolbar to get help on the currently active tool.

### How to Save and Open GeoGebra Files

Saving GeoGebra Files

- Open the File menu and select Save.
- Select the folder GeoGebra Introduction in the appearing dialog window.

- Type in a name for your GeoGebra file. Click Save in order to finish this process.

A file with the extension .ggb is created. This extension identifies GeoGebra files and indicates that they can only be opened with GeoGebra.

Name your files properly: Avoid using spaces or special symbols in a file name since they can cause unnecessary problems when transferred to other computers. Instead, you can use underscores or upper case letters within the file name (e.g. First_Drawing.ggb).

### 16.1.2 Opening GeoGebra Files

- Open a new GeoGebra window (menu File - New window).
- Open a blank GeoGebra interface within the same window (menu File - New).
- Open an already existing GeoGebra file (menu File - Open).
  (i) Navigate through the folder structure in the appearing window.
  (ii) Select a GeoGebra file (extension.ggb) and click Open.

If you did not save the existing construction, yet GeoGebra will ask you to do so before the blank screen / new file is opened.

### 16.1.3 Exploring Polynomials

Preparations

- Open a new GeoGebra window
- Switch to Perspectives - Algebra & Graphics

Construction Steps

| |
|---|
| 1 Enter the cubic polynomial $f(x) = 0.5x^3 + 2x^2 + 0.2x - 1$. |
| 2 Create the roots of polynomial $f$: R = Root[$f$]<br>Hint: If there are more than one root GeoGebra will produce indices for their names if you type in R = (e.g. R1, R2, R3). |
| 3 Create the extrema of polynomial $f$: E = Extremum[$f$]. |
| 4 Create the turning points of polynomial $f$: E = Turning Point[$f$]. |
| 5 Create tangents to $f$ in E1 and E2. |
| 6 Create the inflection point of polynomial $f$: I = Inflection Point[$f$]. |

You might want to change properties of objects (e.g. color of points, style of the tangents, show name and value of the function).

### 16.1.4 Visualizing a System of Linear Equations

Preparations

- Open a new GeoGebra window

- Switch to Perspectives - Algebra & Graphics and show the Grid.

Construction Steps

| |
|---|
| 1 Create slider m_1 with the default settings for sliders.<br>Hint: m_1 gives you m1. |
| 2 Create slider b_1 with the default settings for sliders. |
| 3 Create the linear equation line_1: y = m_1 x + b_1. |
| 4 Create slider m_2 using the default settings for sliders. |
| 5 Create slider b_2 using the default settings for sliders. |
| 6 Create linear equation line_2: y = m_2 x + b_2. |
| 7 Create the dynamic text1: Line 1: and select line_1 from Objects. |
| 8 Create the dynamic text2: Line 2: and select line_2 from Objects. |
| 9 Construct the intersection point A of both lines line1 and line2.<br>Hint: You could use command Intersect [line_1, line_2] instead. |
| 10 Define x coordinate = x(A).<br>Hint: x(A) gives you the x-coordinate of point A. |
| 11 Define y coordinate = y(A).<br>Hint: y(A) gives you the y-coordinate of point A. |
| 12 Create the dynamic text3: Solution: x = and select x coordinate from Objects. |
| 13 Create the dynamic text4: y = and select y coordinate from Objects. |
| 14 Fix the text and sliders so they cant be moved accidentally. |

### 16.1.5 Challenge

Create a similar construction that allows for visualizing the graphical solution of a system of quadratic polynomials.

Hint: Functions need to be entered using the syntax $f(x) = ...$

Note: Such a dynamic figure can also be used to visualize an equation in one variable by entering each side of the equation as one of the two functions.

### Interactive Worksheets

| | |
|---|---|
| The Linear Function $f(x) = mx + c$ | - https://www.geogebra.org/m/s3jEhYHw |
| The Quadratic Function $f(x) = ax^2 + bx + c$ | - https://www.geogebra.org/m/ZBYUrerX |
| The Exponential Function $f(x) = c \cdot a^{kx} + d$ | - https://www.geogebra.org/m/s3jEhYHw |
| The Logarithmic Function $f(x) = c \cdot \log_a(x - k) + d$ | - https://www.geogebra.org/m/EUZVy9Br |
| The Sine Function $f(x) = a \sin(bx + c)$ | - https://www.geogebra.org/m/sw4J5W8D |
| The Sine Function $f(x) = a \sin(bx + c) + d$ | - https://www.geogebra.org/m/rY3Vycsa |
| The Cosine Function $f(x) = a \cos(bx + c)$ | - https://www.geogebra.org/m/azs63teb |
| The Cosine Function $f(x) = a \cos(bx + c) + d$ | - https://www.geogebra.org/m/hxsqX7DF |

## 16.1 EXERCISES

I. Answer the following questions:

(1) Using GeoGebra or any appropriate software, sketch the graph of the following linear equations and write the intercepts.

(a) $2x - 6y = 18$

(b) $3x - 2y = 12$

(c) $-5x + 2y = 10$

(2) Using GeoGebra or any appropriate software, sketch the graph of the following quadratic functions and write its vertex, axis of symmetry and minimum value.

(a) $x^2 + 2x + 8$

(b) $-2x^2 - 7x + 4$

(c) $3x^2 - 10x - 8$

(3) Using GeoGebra or any appropriate software, sketch the graph of the following system of equation and find the solution if exist.

(a) $2x + y = 9$; $3x - y = 16$

(b) $x–2y = –9$; $x + 3y = 16$

(c) $2x–y = 9$; $3x + 4y = –14$

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